TWO-COLOR SOERGEL CALCULUS AND SIMPLE TRANSITIVE 2-REPRESENTATIONS

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ABSTRACT. In this paper we complete the ADE-like classification of simple transitive 2-representations of Soergel bimodules in finite dihedral type, under the assumption of gradeability. In particular, we use bipartite graphs and zigzag algebras of ADE type to give an explicit construction of a graded (non-strict) version of all these 2-representations.

Moreover, we give simple combinatorial criteria for when two such 2-representations are equivalent and for when their Grothendieck groups give rise to isomorphic representations.

Finally, our construction also gives a large class of simple transitive 2-representations in infinite dihedral type for general bipartite graphs.

CONTENTS

1. INTRODUCTION

An essential problem in classical representation theory is the classification of the simple representations of any given algebra, i.e. the parametrization of their isomorphism classes and the explicit construction of a representative of each class.

In 2-representation theory, the actions of algebras on vector spaces are replaced by functorial actions of 2-categories on certain additive or abelian 2-categories. The Grothendieck group of a 2-representation is a classical representation. One can say that the 2-representation decategorifies to the classical representation, or that the latter is a decategorification of the former. Vice versa, one can also say that the 2-representation categorifies the classical representation to which it decategorifies, or that it is a categorification. Note that, in general, categorifications need not be unique.

Examples are 2-representations of the 2-categories which categorify representations of quantum groups, due to (Chuang–)Rouquier and Khovanov–Lauda, and 2-representations of the 2-category of Soergel bimodules, which categorify representations of Hecke algebras.

Mazorchuk–Miemietz [MM16b] defined an appropriate 2-categorical analogue of the simple representations of finite-dimensional algebras, which they called simple transitive 2-representations (of finitary 2-categories). The problem is that their classification is very hard and not well understood in general – except when certain specific conditions are satisfied, as for Soergel bimodules in type A [MM16b, Theorem 21] for example.

The authors of [KMMZ16] studied the so-called small quotient of Soergel bimodules and their simple transitive 2-representations, for all finite Coxeter types. These 2-representations are given by categories on which the bimodules act by endofunctors and the bimodule maps by natural transformations. Each of these categories is equivalent to the (projective or abelian) module category over the path algebra of a finite quiver, which can be obtained by doubling a certain Dynkin diagram. An almost complete classification was given in [KMMZ16], which we now recall.

In every finite Coxeter type of rank strictly greater than two, all the simple transitive 2-representations are equivalent to Mazorchuk and Miemietz's categorification of the cell representations of Hecke algebras, the so-called cell 2-representations.

The rank two case is more delicate. In type $\mathrm{I}_2(n)$, for any $n \in \mathbb{Z}_{>1}$, there is one cell 2-representation of rank one and two higher rank cell 2-representations, which correspond to the two possible bipartitions of the Dynkin diagram of type A_{n-1}. (Here we distinguish a bipartition of a given graph from the opposite bipartition.) When n is odd, these exhaust all simple transitive 2-representations, up to equivalence.

When $n = 2, 4$, it was already known that the same holds, see [Zim17]. However, when n is even and greater than four, it was shown in [KMMZ16] that there exist additional simple transitive 2-representations which are *not* equivalent to cell 2-representations. If one has $n \notin \{12, 18, 30\}$, then there exist exactly two which correspond to the two possible bipartitions of a type $\mathrm{D}_{\frac{n}{2}+1}$ Dynkin diagram.

For $n = 12, 18, 30$, the possible existence of one more additional pair of inequivalent simple transitive 2-representations was discovered, but not proved (not even conjectured) in [KMMZ16]. Should they exist, their underlying Dynkin diagrams were shown to be of type E_6 for $n = 12$, of type E_7 for $n = 18$, and of type E_8 for $n = 30$.

The simple transitive 2-representations with quivers of type A and D were constructed intrinsically in [KMMZ16]. The ones of type A are equivalent to the aforementioned higher rank cell 2-representations, due to Mazorchuk–Miemietz [MM11], and can be constructed as subquotients of the 2-category of Soergel bimodules, as in any finite Coxeter type. The type D simple transitive 2-representations of $\mathrm{I}_2(n)$, for $n > 5$ even, were constructed in [KMMZ16] using an involution on the cell 2-representations, mimicking a construction in [MM17]. Both constructions do not take into account the $\mathbb{Z}$-grading of the Soergel bimodules, but the 2-representations admit a unique compatible grading.

In this paper, we construct *all* graded simple transitive 2-representations of the small quotient of the Soergel bimodules of type $\mathrm{I}_2(n)$ by *different* means. We use Elias' [Eli16] diagrammatic version of the latter 2-category, the so-called two-color Soergel calculus. More precisely, given a Dynkin diagram of type A, D or E with a bipartition, we define two degree-preserving, self-adjoint endofunctors $\boldsymbol{\Theta}_s$ and $\boldsymbol{\Theta}_t$ on the module category over the corresponding quiver – which is a zigzag algebra of type ADE – and, for each generating diagram in the two-color Soergel calculus, a natural transformation between composites of them such that all diagrammatic relations are preserved.

Moreover, we show that two graded simple transitive 2-representations are equivalent if and only if the corresponding bipartite graphs are isomorphic (as bipartite graphs). Finally, we also determine when graded simple transitive 2-representations decategorify to isomorphic representations of the corresponding Hecke algebra, using a purely graph-theoretic property.

Let us give some interesting consequences of our results. First, there are two inequivalent graded simple transitive 2-representations of type E_6 (or E_8) for the Soergel bimodules of type $I_2(12)$ (or $I_2(30)$), which decategorify to isomorphic representations of the associated Hecke algebra. (The two simple transitive 2-representations of type E_7 for the type $I_2(18)$ Soergel bimodules have non-isomorphic decategorifications.) To the best of our knowledge these are the first examples of simple transitive 2-representations of the *same* 2-category which decategorify to isomorphic representations.

Our construction also gives graded simple transitive 2-representations of the 2-category defined by the two-color Soergel calculus in type $I_2(\infty)$ for any bipartite graph, not just the ones of ADE type. (Note that this 2-category is not always isomorphic to the 2-category of Soergel bimodules in type $I_2(\infty)$, cf. Remark 2.18.) For these 2-representations, all of the above statements are still valid.

Potential further developments. We hope that our construction will also be helpful for the construction of simple transitive 2-representations of other 2-categories, e.g. Soergel bimodules in other finite Coxeter types with more 2-cells.

Furthermore, since our preprint first appeared, we wrote a joint paper with Mazorchuk–Miemietz [MMMT16] in which we explain the relation between the simple transitive 2-representations of the Soergel bimodules in finite dihedral type and those of the semisimplified subquotient of the module category of quantum $\mathfrak{sl}_2$ at a root of unity, due to Kirillov–Ostrik [KJO02], [Ost03] and others. This relation is based on Elias' algebraic quantum Satake equivalence [Eli16], [Eli17]. There should be a similar story for $\mathfrak{sl}_{>2}$, and we hope that our paper will help to develop it. (See [MMMT18] for some first steps in this direction.)

And most of our constructions work over certain integral rings, see (4.1). So perhaps our work will also be useful for 2-representation theory in finite characteristic.

Structure of the paper.

(i) In Section 2 we give the details of our graph-theoretical construction.

(ii) In Section 3 we explain some notions concerning 2-categories and 2-representations, focusing on the graded and weak setups (by extending Mazorchuk and Miemietz's constructions).

(iii) In Section 4 we recall Elias' two-color Soergel calculus and define our 2-action of it. Hereby we follow – and generalize – ideas from [AT17] and [KS02].

(iv) Finally, Section 5 contains all proofs.

Remark 1.1. We use colors in this paper, but they are not necessary to read the paper and just a service to the reader. Nevertheless, three colors should be mentioned, i.e. sea-green and tomato denote the two different generators s and t of the dihedral groups, while dark orchid denotes notions which play the role of a dummy and can be replaced by either of the two. Moreover, our notation is designed so that these colors can be distinguished in black-and-white:

(i) The color sea-green is always either accompanied by the symbol s, the associated notions are underlined or we use $\bullet$.

(ii) The color tomato is always either accompanied by the symbol t, the associated notions are overlined or we use $\blacksquare$.

(iii) The color dark orchid is always either accompanied by the symbols u, v or w, the associated notions are neither under- nor overlined or we use $\blacklozenge$. $\blacktriangle$

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2. Bipartite graphs and dihedral 2-representations

In this section we state our main results (see Section 2.4), and provide the background needed to understand them.

2.1. Combinatorics of dihedral groups.

First, we recall some basic notions concerning the dihedral groups.

2.1.1. *The dihedral group and its associated Hecke algebra.* We follow [Eli16] with our conventions. Let

$$\mathrm{W}_n = \langle s,t \mid s^2 = t^2 = 1, s_n = \underbrace{\dots sts}_{n} = \underbrace{\dots tst}_{n} = t_n \rangle \quad \text{and} \quad \mathrm{W}_\infty = \langle s,t \mid s^2 = t^2 = 1 \rangle$$

be the *dihedral groups* of order $2n \in \mathbb{Z}_{>0}$ and of infinite order respectively, presented as the rank two Coxeter groups of type $\mathrm{I}_2(n)$ and $\mathrm{I}_2(\infty)$. When no confusion is possible, we write W for either W_n or W_∞. The two generators s and t are always sea-green s and tomato t colored. (Throughout, we allow $n = 1$ which is to be understood by dropping one color, say tomato t, and all notions involving it.)

Here and in the following, we denote by s_k respectively by t_k a sequence of length k, alternating in s and t with rightmost symbol s respectively t. In general, we write an element of W as a finite word $w = w_l \cdots w_1$ with $w_k \in \{s,t\}$ (including the empty word $\emptyset$). We say that the word is reduced if it is equal to s_k or t_k for some $k \in \mathbb{Z}_{\geq 0}$. Throughout the remainder of this paper, when we write $w \in \mathrm{W}$ as a word, we always assume that the word is reduced if not stated otherwise.

Moreover, let H_n or H_∞ denote the associated *Hecke algebra* over $\mathbb{C}(\mathrm{v})$, with v being an indeterminate. Again, when no confusion is possible, we write H for H_n or H_∞. This algebra has generators T_s and T_t, and relations

$$T_s^2 = (\mathrm{v}^{-2} - 1) \cdot T_s + \mathrm{v}^{-2}, \qquad T_t^2 = (\mathrm{v}^{-2} - 1) \cdot T_t + \mathrm{v}^{-2}, \qquad T_{s_n} = T_{t_n}.$$

For any $w \in \mathrm{W}$, pick a reduced word $w_l \cdots w_1$ which represents it. Then the element $T_w \in \mathrm{H}$ denotes the product $T_{w_l} \cdots T_{w_1}$ and $T_\emptyset = 1$. The element T_w does not depend on the choice of the reduced word and $\{T_w \mid w \in \mathrm{W}\}$ forms a basis of H. So for $\mathrm{v} = 1$ (when working over $\mathbb{C}[\mathrm{v}, \mathrm{v}^{-1}]$) we recover $\mathbb{C}[\mathrm{W}]$.

2.1.2. *The dihedral Kazhdan–Lusztig basis.* Denote by $\ell(w)$ the length of $w \in \mathrm{W}$ and by $\leq$ the Bruhat order on W. For any $w \in \mathrm{W}$ define

$$\theta_w = \mathrm{v}^{\ell(w)} \cdot \textstyle\sum_{w' \leq w} T_{w'}, \quad w, w' \in \mathrm{W}.$$

The set $\{\theta_w \mid w \in \mathrm{W}\}$ forms a basis of H, called the *Kazhdan–Lusztig basis*.

For an expression $w = w_l \cdots w_1$ for $w \in \mathrm{W}$ – not necessarily reduced – let $\theta_{\overline{w}} = \theta_{w_l} \cdots \theta_{w_1}$. This element does depend on the choice of expression for w, even amongst reduced expressions. Choosing one reduced expression for each $w \in \mathrm{W}$, we get another basis $\{\theta_{\overline{w}} = \theta_{w_{\ell(w)}} \cdots \theta_{w_1} \mid w \in \mathrm{W}\}$ for H, called the *Bott–Samelson basis*, cf. Section 4.1. (Note that we write $\theta_{\overline{w}}$ to distinguish the Bott–Samelson from the Kazhdan–Lusztig basis.)

Example 2.1. One has $\theta_s = \mathrm{v}(T_s + 1)$, $\theta_t = \mathrm{v}(T_t + 1)$, and an easy calculation gives $\theta_{sts} = \theta_{\overline{sts}} - \theta_s = \theta_s\theta_t\theta_s - \theta_s$. In general, $\theta_w = \theta_{\overline{w}} \mp$ "lower order terms". ▲

For any $n \in \mathbb{Z}_{>0}$, the group W_n has a unique longest element

$$w_0(n) = w_0 = s_n = t_n.$$

(In case "$n = \infty$" there is no such w_0.) In this paper, we only consider categorifications of H_n-representations which are killed by $\theta_{w_0(n)} = \theta_{w_0}$. In fact, the only decategorification of a simple transitive 2-representation of H_n which is not killed by θ_{w_0} is, by [MM16b, Theorem 18], the trivial H_n-representation, cf. Theorem III.

2.1.3. *The defining relations satisfied by the Kazhdan–Lusztig basis elements.* Define recursively the integers d_l^k via:

$$\begin{aligned} d_1^1 = 1, \quad d_l^k = 0, \quad & \text{unless } 0 < k \leq l \text{ and } l - k \text{ is even,} \\ d_l^k = {} & d_{l-1}^{k-1} - d_{l-2}^k. \end{aligned} \tag{2.1}$$

Let $[2]_{\mathrm{v}} = \mathrm{v} + \mathrm{v}^{-1}$. When considering the basis $\{\theta_{\overline{w}} \mid w \in \mathrm{W}\}$, the defining relations of H – by e.g. [Eli16, Section 2.2] – are

$$\theta_s\theta_s = [2]_{\mathrm{v}} \cdot \theta_s, \qquad \theta_t\theta_t = [2]_{\mathrm{v}} \cdot \theta_t, \tag{2.2}$$

$$\textstyle\sum_{k\in\mathbb{Z}_{\geq 0}} d_n^k \cdot \theta_{\overline{s_k}} = \sum_{k\in\mathbb{Z}_{\geq 0}} d_n^k \cdot \theta_{\overline{t_k}}. \tag{2.3}$$

Note that either sum in (2.3) is equal to θ_{w_0}.

Example 2.2. The first few non-zero numbers from (2.1) are as follows.

$$\begin{array}{cc|ccccccc} & & & & & k & & & \\ & & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \hline & 1 & 1 & & & & & & \\ & 2 & & 1 & & & & & \\ & 3 & -1 & & 1 & & & & \\ l & 4 & & -2 & & 1 & & & \\ & 5 & 1 & & -3 & & 1 & & \\ & 6 & & 3 & & -4 & & 1 & \\ & \vdots & & & \ddots & & \ddots & & \ddots \end{array}.$$

Hence, if $n = l = 3$, then (2.3) gives

$$\theta_s\theta_t\theta_s - \theta_s = \theta_t\theta_s\theta_t - \theta_t \ (= \theta_{w_0}). \tag{2.4}$$

When "$n = \infty$", the only relations are the ones from (2.2). ▲

2.2. **Bipartite graphs.** Next, we recall some basics about bipartite graphs and fix notation which we use throughout.

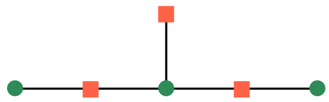
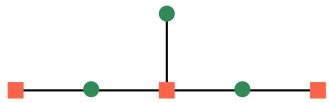
2.2.1. *A reminder on bipartite graphs.* Let G be a connected, unoriented, finite graph without loops and with at most one edge between each pair of vertices. Let $V = V(G)$ be the set of vertices of G and assume that the vertices are numbered. If these numbers are divided into two disjoint subsets I and J, such that

$$V = \underline{\mathrm{S}} \coprod \overline{\mathrm{T}}, \qquad \underline{\mathrm{S}} = \{\underline{\mathrm{i}} \mid i \in I\}, \qquad \overline{\mathrm{T}} = \{\overline{\mathrm{j}} \mid j \in J\},$$

with no edges connecting vertices within each set, then we call the triple $(G, \underline{\mathrm{S}}, \overline{\mathrm{T}})$ a *bipartite graph*. When no confusion is possible, we simply write G for a bipartite graph. Note that a bipartition of G is the same as a two-coloring of its vertices.

Two bipartite graphs $(G, \underline{\mathrm{S}}, \overline{\mathrm{T}})$ and $(G', \underline{\mathrm{S}}', \overline{\mathrm{T}}')$ are called *isomorphic* if there is an isomorphism of graphs between G and G' which sends $\underline{\mathrm{S}}$ to $\underline{\mathrm{S}}'$ and $\overline{\mathrm{T}}$ to $\overline{\mathrm{T}}'$.

Example 2.3. One crucial example in this paper is the type E_6 graph:

$G =$  , $G' =$  .

These two-colorings give non-isomorphic bipartite graphs of type E_6. As we will see later, these will give rise to two inequivalent 2-representations categorifying the same H-module, see Example 5.10 (keeping Example 2.4 in mind). ▲

We write $\underline{\mathrm{i}}{-}\overline{\mathrm{j}} (= \overline{\mathrm{j}}{-}\underline{\mathrm{i}})$ in case $\underline{\mathrm{i}}$ and $\overline{\mathrm{j}}$ are connected in G.

2.2.2. *Adjacency matrices and spectra.* Given any graph G, the *adjacency matrix* $A(G)$ of G is the symmetric $|V| \times |V|$-matrix whose only non-zero entries are $A(G)_{\underline{\mathrm{i}},\overline{\mathrm{j}}} = 1$ for $\underline{\mathrm{i}}{-}\overline{\mathrm{j}}$. The *spectrum* S_G of G is the multiset of all eigenvalues of $A(G)$ (which are all real), repeating each one of them according to its multiplicity.

Since we assume G to be bipartite, we can clearly choose an ordering of the vertices (which we will always do from now on) such that

$$A(G) = \begin{pmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{pmatrix} \tag{2.5}$$

for some matrix A of size $|\underline{\mathrm{S}}| \times |\overline{\mathrm{T}}|$. Next, recall that S_G is a symmetric set (see e.g. [BH12, Proposition 3.4.1]), and the fact from linear algebra that

$$AA^{\mathrm{T}} \text{ has an eigenvalue } \alpha \neq 0 \;\Leftrightarrow\; A^{\mathrm{T}}A \text{ has an eigenvalue } \alpha \neq 0. \tag{2.6}$$

Thus, the non-zero elements (counting multiplicities) of S_G are $\pm\sqrt{\alpha}$ for α as in (2.6).

2.2.3. *Spectrum-color-equivalence.* Given two bipartite graphs G and G'. We call them *spectrum-color-equivalent* if $|\underline{\mathrm{S}}| = |\underline{\mathrm{S}}'|$, $|\overline{\mathrm{T}}| = |\overline{\mathrm{T}}'|$ and $S_G = S_{G'}$, and *spectrum-color-inequivalent* otherwise. This clearly gives rise to an equivalence relation.

Example 2.4. Take the two graphs G and G' from Example 2.3. Then G and G' are non-isomorphic as bipartite graphs, but they are spectrum-color-equivalent.

More generally, any bipartite graph with $|\underline{\mathrm{S}}| = |\overline{\mathrm{T}}|$ is spectrum-color-equivalent to the bipartite graph with the opposite two-coloring. ▲

2.3. **Quivers and categorical representations of dihedral groups.** Fix a bipartite graph G. The *double quiver* Q_G associated to G is the oriented graph obtained from G by doubling each edge and giving opposite orientations to the two resulting edges, called arrows. Such an arrow is denoted by $\overline{\mathrm{j}}|\underline{\mathrm{i}}$ if it starts at $\underline{\mathrm{i}}$ and ends at $\overline{\mathrm{j}}$, and by $\underline{\mathrm{i}}|\overline{\mathrm{j}}$ if it starts at $\overline{\mathrm{j}}$ and ends at $\underline{\mathrm{i}}$ (i.e. we are using the "operator notation").

Definition 2.5. Two distinct arrows of Q_G are called *partners* if they come from the same edge in G. ▲

Thus, each arrow has precisely one partner pointing in the opposite direction.

Example 2.6. An example of a double-quiver is:

$$G=\ \underline{1}\text{——}\overline{2}\text{——}\underline{3}\ \rightsquigarrow\ Q_G=\ \underline{1}\rightleftarrows\overline{2}\rightleftarrows\underline{3}\ .$$

In the above example, $\overline{2}|\underline{1}$ and $\underline{1}|\overline{2}$ are partners, and so are $\underline{3}|\overline{2}$ and $\overline{2}|\underline{3}$. ▲

As already pointed out, we will see that only bipartite graphs of ADE type give rise to 2-representations of Soergel bimodules in finite dihedral type.

2.3.1. *The path algebra associated to a bipartite graph.* We work over certain base rings $\mathscr{A}_{\mathrm{gen}}$, $\mathscr{A}_n$ or $\mathscr{A}_{G,\vec{\lambda},\mathrm{q}}$, all of which are subrings of $\mathbb{C}$. We will impose some technical conditions on these which we discuss in Remark 4.1, and the reader should think of these as playing the role of an integral form.

If no confusion can arise, we simply write $\mathscr{A}$ for either $\mathscr{A}_{\mathrm{gen}}$, $\mathscr{A}_n$ or $\mathscr{A}_{G,\vec{\lambda},\mathrm{q}}$. Of course, we can always extend the scalars to $\mathbb{C}$ if necessary.

Let $\mathrm{P}G$ be the path algebra of Q_G over $\mathscr{A}$, such that multiplication on the left is given by post-composition, on the right by pre-composition. We denote the multiplication by $\bullet$ and consider $\mathrm{P}G$ to be graded by the path length.

By convention, $\underline{\mathrm{i}}$ and $\overline{\mathrm{j}}$ denote the corresponding paths of length zero. Paths of length one are in one-to-one correspondence with arrows of Q_G, and we call them arrows too.

Definition 2.7. Let $\mathrm{Q}G$ denote the quotient algebra obtained from $\mathrm{P}G$ by the following relations.

▷ **The two and three steps relations.**

(2.QG1) The composite of two arrows is zero unless they are partners.

(2.QG2) The composite of three arrows is zero.

▷ **All non-zero partner composites are equal.** Assume that $\underline{\mathrm{i}}\text{—}\overline{\mathrm{j}}_a$, for $a=1,\dots,b$. Then:

$$\underline{\mathrm{i}}|\overline{\mathrm{j}}_1\bullet\overline{\mathrm{j}}_1|\underline{\mathrm{i}}=\cdots=\underline{\mathrm{i}}|\overline{\mathrm{j}}_b\bullet\overline{\mathrm{j}}_b|\underline{\mathrm{i}}=\underline{\mathrm{i}}|\underline{\mathrm{i}},\tag{2.QG3}$$

where the path $\underline{\mathrm{i}}|\underline{\mathrm{i}}$, called *loop*, is defined by the above equations.

Similar relations hold with $\underline{\mathrm{i}}$ and $\overline{\mathrm{j}}$ swapped.

(Note that the relation (2.QG2) is a consequence of (2.QG1) and (2.QG3) as long as G has two or more edges.) We call $\mathrm{Q}G$ the *type G-quiver algebra.* ▲

The defining relations of $\mathrm{Q}G$ are homogeneous, so $\mathrm{Q}G$ inherits the path length grading. Clearly, the primitive idempotents of $\mathrm{Q}G$ are the i's.

Example 2.8. The relations (2.QG1), (2.QG2) and (2.QG3) might be familiar to readers who know about the so-called zigzag algebras in the spirit of [HK01].

Let us give two examples. Fix $m\in\mathbb{Z}_{\geq 0}$. Let G be a Dynkin graph of type A_m or $\tilde{\mathrm{A}}_{2m-1}$ with m, respectively $2m$ vertices. Then the associated Q_G's are of the following form (if we consider the evident two-coloring of G).

$$\underline{1}\rightleftarrows\overline{2}\rightleftarrows\underline{3}\rightleftarrows\cdots\rightleftarrows \mathrm{m{-}2}\rightleftarrows\mathrm{m{-}1}\rightleftarrows\mathrm{m}\ ,\qquad
\begin{tikzcd}[column sep=small]
 & \overline{1} \arrow[r, shift left] \arrow[dl, shift left] & \underline{2} \arrow[l, shift left] \arrow[dr, shift left] & \\
\underline{0} \arrow[ur, shift left] \arrow[dr, shift left] & & & \overline{3} \arrow[ul, shift left] \arrow[dl, shift left] \\
 & \overline{2\mathrm{m}{-}1} \arrow[ul, shift left] \ \cdots & \underline{4} \arrow[ur, shift left] &
\end{tikzcd}\ .$$

The defining relations of the associated quiver algebras are of the form

$$(2.\mathrm{QG1})\colon\quad \mathrm{i{+}2|i{+}1}\circ\mathrm{i{+}1|i} = 0 = \mathrm{i{-}2|i{-}1}\circ\mathrm{i{-}1|i},$$
$$(2.\mathrm{QG3})\colon\quad \mathrm{i|i{+}1}\circ\mathrm{i{+}1|i} = \mathrm{i|i} = \mathrm{i|i{-}1}\circ\mathrm{i{-}1|i},$$

with indices modulo $2m$ in the cyclic case.

The quiver algebra for the Dynkin graph of type A_m will be of importance later on and we denote it by QA_m. Similarly, we denote by $\mathrm{Q}\tilde{\mathrm{A}}_{2m-1}$ its cyclic counterpart. ▲

In the above and throughout; for $m = 0$ and $m = 1$ we let $\mathrm{Q}G = \mathrm{QA}_0 = \mathscr{A}$ and $\mathrm{Q}G = \mathrm{QA}_1 \cong \mathscr{A}[X]/(X^2)$, by convention.

Remark 2.9. The algebras from Example 2.8 also appear in the context of categorical braid group actions, e.g. in [GTW17] and [KS02]. Note that Khovanov–Seidel have the additional relation $\underline{1}|\overline{2}|\underline{1} = 0$. ▲

2.3.2. *Some bimodules.* Let us denote by P_{i} and ${}_{\mathrm{i}}P$ the left and the right ideal of $\mathrm{Q}G$ generated by i, respectively. They are clearly graded $\mathrm{Q}G$-modules.

Example 2.10. We can easily visualize them for QA_m with $1 < i < m$ as

$$(2.10)\qquad P_{\mathrm{i}} = \begin{tikzcd} {} & \mathrm{i} \arrow[l, "\mathrm{i-1|i}"'] \arrow[r, "\mathrm{i+1|i}"'] \arrow[loop above, "\mathrm{i|i}"] & {} \end{tikzcd}, \qquad {}_{\mathrm{i}}P = \begin{tikzcd} {} \arrow[r, "\mathrm{i|i-1}"'] & \mathrm{i} \arrow[loop above, "\mathrm{i|i}"] & {} \arrow[l, "\mathrm{i|i+1}"'] \end{tikzcd}.$$

They have basis vectors $\mathrm{i}, \mathrm{i{-}1|i}, \mathrm{i{+}1|i}, \mathrm{i|i}$ and $\mathrm{i}, \mathrm{i|i{-}1}, \mathrm{i|i{+}1}, \mathrm{i|i}$, respectively, of degree $0, 1, 1, 2$. So they are free of $\mathscr{A}$-rank three for any $m \neq 0, 1$. ▲

Because the i form a complete set of orthogonal primitive idempotents in $\mathrm{Q}G$, the P_{i} and the ${}_{\mathrm{i}}P$ are graded, indecomposable projective modules and all graded, indecomposable left – respectively right – $\mathrm{Q}G$-modules are of the form $P_{\mathrm{i}}\{\mathtt{a}\}$, respectively ${}_{\mathrm{i}}P\{\mathtt{a}\}$, for some shift $\mathtt{a} \in \mathbb{Z}$. Let $\otimes = \otimes_{\mathscr{A}}$ be the tensor product over $\mathscr{A}$. Then $P_{\mathrm{i}}\{-1\} \otimes {}_{\mathrm{i}}P$ is clearly a graded $\mathrm{Q}G$-bimodule. These bimodules will be important in this paper.

2.3.3. *Endofunctors associated to bipartite graphs.* We denote by $\mathbf{G}^{\mathrm{gr}} = \mathrm{Q}G\text{-p}\mathbf{Mod}^{\mathrm{gr}}$ the category of graded projective, (left) $\mathrm{Q}G$-modules, which are free of finite $\mathscr{A}$-rank. Our next goal, following [KS02, Section 2] and [AT17, Section 3], is to define endofunctors

$$\boldsymbol{\mathcal{V}}_{\mathrm{i}}\colon \mathbf{G}^{\mathrm{gr}} \to \mathbf{G}^{\mathrm{gr}}, \quad i \in G.$$

Let $\widehat{\otimes} = \otimes_{\mathrm{Q}G}$ be the tensor product over $\mathrm{Q}G$. We define

$$\boldsymbol{\mathcal{V}}_{\mathrm{i}}(X) = P_{\mathrm{i}}\{-1\} \otimes {}_{\mathrm{i}}P \,\widehat{\otimes}\, X, \qquad \boldsymbol{\mathcal{V}}_{\mathrm{i}}(f) = \mathrm{ID}_{P_{\mathrm{i}}\{-1\}\otimes{}_{\mathrm{i}}P} \,\widehat{\otimes}\, f.$$

Here $X, Y \in \mathbf{G}^{\mathrm{gr}}$ and $f \in \mathrm{Hom}_{\mathbf{G}^{\mathrm{gr}}}(X, Y)$. (Since $P_{\mathrm{i}}\{-1\} \otimes {}_{\mathrm{i}}P$ is clearly *biprojective*, i.e. projective both as a left and as a right $\mathrm{Q}G$-module, the functor $\boldsymbol{\mathcal{V}}_{\mathrm{i}}$ sends graded projectives to graded projectives.) As in [AT17, Section 3.3] we immediately obtain

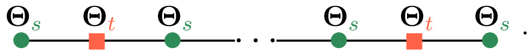
$$(2.11)\qquad \boldsymbol{\mathcal{V}}_{\mathrm{i}}(P_{\mathrm{j}}) \cong \begin{cases} P_{\mathrm{i}}\{-1\} \oplus P_{\mathrm{i}}\{+1\}, & \text{if } \mathrm{i} = \mathrm{j}, \\ P_{\mathrm{i}}, & \text{if } \mathrm{i}{-}\mathrm{j}, \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.11. By looking at (2.10) one observes that ${}_{\mathrm{i}}P \,\widehat{\otimes}\, P_{\mathrm{i}} \cong \mathscr{A}(\mathrm{i}) \oplus \mathscr{A}(\mathrm{i|i})$. This isomorphism is given by multiplication of paths. Consequently, the degree two $\mathrm{Q}G$-endomorphism $\boldsymbol{\mathcal{V}}_{\mathrm{i}}(\mathrm{i|i})$ of $P_{\mathrm{i}}\{-1\} \oplus P_{\mathrm{i}}\{+1\}$ sends the copy $P_{\mathrm{i}}\{-1\}$ identically to $P_{\mathrm{i}}\{+1\}$, and is zero elsewhere (recall that $\mathrm{i|i}\circ\mathrm{i|i} = 0$). ▲

Following [AT17, Section 3.3] we define

$$\boldsymbol{\Theta}_{s} = \bigoplus\nolimits_{\underline{\mathrm{i}}\in G}\boldsymbol{\mathcal{V}}_{\underline{\mathrm{i}}}, \qquad \boldsymbol{\Theta}_{t} = \bigoplus\nolimits_{\overline{\mathrm{j}}\in G}\boldsymbol{\mathcal{V}}_{\overline{\mathrm{j}}}.$$

Example 2.12. We sum over the graph of type A_m as (in case m is odd):



Note that the opposite two-coloring of G switches $\mathbf{\Theta}_s$ and $\mathbf{\Theta}_t$. In general, this switch need not be a natural isomorphism. ▲

By using (2.11), one directly checks that

$$\mathbf{\Theta}_s(P_{\mathrm{i}}) \cong \begin{cases} P_{\underline{\mathrm{i}}}\{-1\} \oplus P_{\underline{\mathrm{i}}}\{+1\}, & \text{if } i \in \underline{\mathrm{S}}, \\ \bigoplus_{\underline{\mathrm{j}} - \overline{\mathrm{i}}} P_{\underline{\mathrm{j}}}, & \text{if } i \in \overline{\mathrm{T}}, \end{cases} , \mathbf{\Theta}_t(P_{\mathrm{i}}) \cong \begin{cases} P_{\overline{\mathrm{i}}}\{-1\} \oplus P_{\overline{\mathrm{i}}}\{+1\}, & \text{if } i \in \overline{\mathrm{T}}, \\ \bigoplus_{\underline{\mathrm{i}} - \overline{\mathrm{j}}} P_{\overline{\mathrm{j}}}, & \text{if } i \in \underline{\mathrm{S}}. \end{cases} \tag{2.12}$$

2.3.4. *Dihedral modules associated to bipartite graphs.*

Definition 2.13. Let G^{v} be the $\mathbb{C}(\mathrm{v})$-vector space on the basis $\{\mathrm{i} \mid i \in G\}$. Define a H_∞-action on G^{v} via

$$\theta_s \cdot \underline{\mathrm{i}} = [2]_{\mathrm{v}} \cdot \underline{\mathrm{i}}, \quad \theta_s \cdot \overline{\mathrm{i}} = \textstyle\sum_{\underline{\mathrm{j}} - \overline{\mathrm{i}}} \underline{\mathrm{j}}, \quad \theta_t \cdot \overline{\mathrm{i}} = [2]_{\mathrm{v}} \cdot \overline{\mathrm{i}}, \quad \theta_t \cdot \underline{\mathrm{i}} = \textstyle\sum_{\underline{\mathrm{i}} - \overline{\mathrm{j}}} \overline{\mathrm{j}}. \tag{2.13}$$

(The reader is encouraged to verify that this gives indeed a H_∞-module structure.) We denote the associated algebra homomorphism by $\mathrm{G}\colon \mathrm{H}_\infty \to \mathrm{G}^{\mathrm{v}}$. ▲

Let $[\mathbf{G}^{\mathrm{gr}}]_{\mathbb{C}(\mathrm{v})} = K_0^{\oplus}(\mathbf{G}^{\mathrm{gr}}) \otimes_{\mathbb{Z}[\mathrm{v},\mathrm{v}^{-1}]} \mathbb{C}(\mathrm{v})$ denote the split Grothendieck group tensored with the field $\mathbb{C}(\mathrm{v})$. (As usual, the grading shift decategorifies to multiplication by v.) If we denote by $[\cdot]$ a class in $[\mathbf{G}^{\mathrm{gr}}]_{\mathbb{C}(\mathrm{v})}$, then we get a so-called weak categorification (not to be confused with the weak – in the sense of non-strict – setup that we will meet below), as the following proposition shows.

Proposition 2.14. The functors $\mathbf{\Theta}_s$ and $\mathbf{\Theta}_t$ decategorify to $\mathbb{C}(\mathrm{v})$-linear endomorphisms on $[\mathbf{G}^{\mathrm{gr}}]_{\mathbb{C}(\mathrm{v})}$, which thus becomes an H_∞-module. The $\mathbb{C}(\mathrm{v})$-linear map

$$\zeta_{\mathrm{G}}\colon \mathrm{G}^{\mathrm{v}} \xrightarrow{\cong} [\mathbf{G}^{\mathrm{gr}}]_{\mathbb{C}(\mathrm{v})}, \quad \zeta_{\mathrm{G}}(i) = [P_{\mathrm{i}}], \tag{2.14}$$

is an isomorphism of H_∞-modules, intertwining the actions of θ_s and $[\mathbf{\Theta}_s]$, and that of θ_t and $[\mathbf{\Theta}_t]$. □

Proposition 2.15. Let G be of ADE type and n be its Coxeter number, i.e.:

(A) For G of type A_m we let $n = m + 1$.

(D) For G of type D_m we let $n = 2m - 2$.

(E) For G of type E_6 we let $n = 12$, for G of type E_7 we let $n = 18$, and G of type E_8 we let $n = 30$.

Then the H_∞ actions on G^{v} and $[\mathbf{G}^{\mathrm{gr}}]_{\mathbb{C}(\mathrm{v})}$ descend to H_n-actions which are matched by ζ_{G} as in (2.14). (We denote these by $\mathrm{G}_n\colon \mathrm{H}_n \to \mathrm{G}^{\mathrm{v}}$).

These are the only G (with more than one vertex) and n for which this holds. □

Remark 2.16. Proposition 2.15 appears as a special case of [Lus83, Proposition 3.8], and was rediscovered in [KMMZ16, Sections 5, 6 and 7]. Note hereby that Lusztig proves his statement using the combinatorics of cells in Coxeter groups, while [KMMZ16] uses categorical results. In this paper, we give an independent proof using spectral graph theory. ▲

Example 2.17. Write $\mathbf{A}^{\mathrm{gr}}(3) = \mathrm{QA}_3\text{-p}\mathbf{Mod}^{\mathrm{gr}}$ and $\tilde{\mathbf{A}}^{\mathrm{gr}}(3) = \mathrm{Q}\tilde{\mathrm{A}}_3\text{-p}\mathbf{Mod}^{\mathrm{gr}}$. Then $[\mathbf{\Theta}_s]$ and $[\mathbf{\Theta}_t]$ act on $[\mathbf{A}^{\mathrm{gr}}(3)]_{\mathbb{C}(\mathrm{v})}$ and $[\tilde{\mathbf{A}}^{\mathrm{gr}}(3)]_{\mathbb{C}(\mathrm{v})}$ via

$$[\mathbf{\Theta}_s] = \begin{pmatrix} [2]_{\mathrm{v}} & 0 & 1 \\ 0 & [2]_{\mathrm{v}} & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad [\mathbf{\Theta}_t] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & [2]_{\mathrm{v}} \end{pmatrix} \quad (\text{in type } \mathrm{A}_3),$$

$$[\boldsymbol{\Theta}_s] = \begin{pmatrix} [2]_{\mathrm{v}} & 0 & 1 & 1 \\ 0 & [2]_{\mathrm{v}} & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad [\boldsymbol{\Theta}_t] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & [2]_{\mathrm{v}} & 0 \\ 1 & 1 & 0 & [2]_{\mathrm{v}} \end{pmatrix} \quad (\text{in type } \tilde{\mathrm{A}}_3).$$

(These are written on the bases $\{[P_{\underline{1}}],[P_{\underline{3}}],[P_{\overline{2}}]\}$ and $\{[P_{\underline{0}}],[P_{\underline{2}}],[P_{\overline{1}}],[P_{\overline{3}}]\}$.)

These matrices give $[\mathbf{A}^{\mathrm{gr}}(3)]_{\mathbb{C}(\mathrm{v})}$ and $[\tilde{\mathbf{A}}^{\mathrm{gr}}(3)]_{\mathbb{C}(\mathrm{v})}$ the structure of an H_∞-module. Additionally, the relation from (2.3) holds in type A_3 since we have

$$[\boldsymbol{\Theta}_s][\boldsymbol{\Theta}_t][\boldsymbol{\Theta}_s][\boldsymbol{\Theta}_t] - 2\cdot[\boldsymbol{\Theta}_s][\boldsymbol{\Theta}_t] = [\boldsymbol{\Theta}_t][\boldsymbol{\Theta}_s][\boldsymbol{\Theta}_t][\boldsymbol{\Theta}_s] - 2\cdot[\boldsymbol{\Theta}_t][\boldsymbol{\Theta}_s].$$

This shows that $[\mathbf{A}^{\mathrm{gr}}(3)]_{\mathbb{C}(\mathrm{v})}$ has the structure of an H_4-module. ▲

2.4. **Strong dihedral 2-representations.** In Section 3 we will explain how to adapt Marzorchuk and Miemietz's definition of finitary 2-categories, their 2-representations and related notions (see e.g. [MM11], [MM16a] or [MM16b]) to our setup. At this point of the paper it is enough to roughly see them as a "higher analog" of graded, finite-dimensional algebras and their graded, finite-dimensional representations.

By using Proposition 2.14, we can identify $[\mathbf{G}^{\mathrm{gr}}]_{\mathbb{C}(\mathrm{v})}$ with G^{v} as H-modules. The next theorem says that the functorial action of $\boldsymbol{\Theta}_s$ and $\boldsymbol{\Theta}_t$ on $\mathbf{G}^{\mathrm{gr}}$ can be extended to a 2-representation of $\boldsymbol{\mathcal{D}}_\infty$ and/or $\boldsymbol{\mathcal{D}}_n$, the 2-categories – defined by generating 2-morphisms and relations – given by the two-color Soergel calculi, due to Elias [Eli16]. (We will recall them in Section 4.1.)

We use the same base ring $\mathscr{A}$ to define $\boldsymbol{\mathcal{D}}_\infty$ and $\boldsymbol{\mathcal{D}}_n$ as we did for the quiver algebras in Section 2.3. (More details will be given in Section 4.1.) Elias' construction of $\boldsymbol{\mathcal{D}}_\infty$ requires the choice of an invertible element q in the base ring (which is ultimately embedded in the complex numbers). When q is not a root of unity, $\boldsymbol{\mathcal{D}}_\infty$ is equivalent to the 2-category of Soergel bimodules in type $\mathrm{I}_2(\infty)$. When q is a complex, primitive $2n$th root of unity, this is no longer true – cf. Remark 2.18 – but $\boldsymbol{\mathcal{D}}_n$ is equivalent to the 2-category of Soergel bimodules in type $\mathrm{I}_2(n)$.

Theorem I. (**Dihedral 2-actions.**)

(a) For any bipartite graph G, there is at least one value of $\mathrm{q} \in \mathbb{C} - \{0\}$ for which there exists an additive, degree-preserving, $\mathscr{A}$-linear, weak 2-functor (defined in Section 4.2)

$$\boldsymbol{\mathcal{G}}_\infty \colon \boldsymbol{\mathcal{D}}_\infty^\star \to \mathrm{p}\mathbf{End}^\star(\mathbf{G}^{\mathrm{gr}}).$$

(b) If G is as in (A), (D) or (E) and q is a complex, primitive $2n$th root of unity, then $\boldsymbol{\mathcal{G}}_\infty$ gives rise to an additive, degree-preserving, $\mathscr{A}$-linear, weak 2-functor

$$\boldsymbol{\mathcal{G}}_n \colon \boldsymbol{\mathcal{D}}_n^\star \to \mathrm{p}\mathbf{End}^\star(\mathbf{G}^{\mathrm{gr}})$$

such that the following diagram commutes

$$\begin{tikzcd} \mathbf{Kar}(\boldsymbol{\mathcal{D}}_n) \arrow[r, "\widetilde{\boldsymbol{\mathcal{G}}}_n"] \arrow[d, "{[\cdot]_{\mathbb{C}(\mathrm{v})}}"'] & \mathrm{p}\mathbf{End}(\mathbf{G}^{\mathrm{gr}}) \arrow[d, "{[\cdot]_{\mathbb{C}(\mathrm{v})}}"] \\ \mathrm{H}_n \arrow[r, "\mathrm{G}_n"'] & \mathrm{End}(\mathrm{G}^{\mathrm{v}}). \end{tikzcd}$$

These are the only (non-trivial) G's and n's for which this holds. □

Here **Kar** denotes the Karoubi envelope, $\widetilde{\cdot}$ the lift of a functor to the Karoubi envelope and p**End** denotes the 2-category of biprojective endofunctors, see Example 3.2. We will explain the meaning of * in the next section.

Remark 2.18. In ADE type, we need q in the Soergel calculus to be a complex, primitive root of unity. In that case – as already remarked – $\mathcal{D}_\infty$ does not quite categorify the Hecke algebra H_∞, see [Eli16, Remarks 5.31 and 5.32]. However, if one allows infinite ADE type graphs, then one can work with a generic parameter. The blueprint example is the infinite type A graph as considered in [AT17], for example. ▲

For the following two theorems we switch to $\mathbb{C}$ for our ground field, and we keep the parameter q fixed.

Theorem II. (**Equivalences and isomorphisms.**)

(a) All 2-representations as in Theorem I are graded simple transitive 2-representations of $\mathbf{Kar}(\mathcal{D}_\infty)^\star$ respectively of $\mathbf{Kar}(\mathcal{D}_n)^\star$.

(b) Two 2-representations as in Theorem I are equivalent if and only if their bipartite graphs are isomorphic.

(c) Two 2-representations as in Theorem I decategorify to isomorphic H-modules if and only if their bipartite graphs are spectrum-color-equivalent.

(d) All 2-representations as in Theorem I factor through graded simple transitive 2-representations of $\mathbf{Kar}(\mathcal{D}^{\mathrm{f}}_\infty)^\star$ respectively of $\mathbf{Kar}(\mathcal{D}^{\mathrm{f}}_n)^\star$. □

Hereby $^{\mathrm{f}}$ means that we work over the coinvariant algebra.

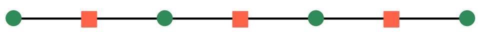
We stress that (c) of Theorem II holds for the analogs from Proposition 2.14 and Proposition 2.15 as well.

With these theorems we can complete the classification from [KMMZ16], where rank means the rank on the level of the Grothendieck groups (for $n = 1$ cf. Example 3.8).

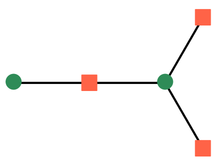
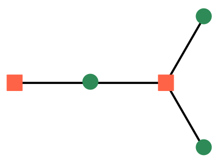
Theorem III. (**Classification.**) There is a bijection between the equivalence classes of graded simple transitive 2-representations of $\mathcal{D}^\star_n$ (of rank > 1) and the isomorphism classes of bipartite graphs as in (A), (D) or (E) with Coxeter number n, for $n \in \mathbb{Z}_{>1}$.

This completes the classification from [KMMZ16], cf. Remark 5.11.

Example 2.19. For $n = 8$, there are four G's of type (A), (D) or (E) which are non-isomorphic as bipartite graphs:

$|\underline{\mathrm{S}}| = 4, |\overline{\mathrm{T}}| = 3$:  ,

$|\underline{\mathrm{S}}| = 3, |\overline{\mathrm{T}}| = 4$:  ,

$|\underline{\mathrm{S}}| = 2, |\overline{\mathrm{T}}| = 3$:  , $|\underline{\mathrm{S}}| = 3, |\overline{\mathrm{T}}| = 2$:  . ▲

In general, we get the following complete list of equivalence classes of higher rank graded simple transitive 2-representations of $\mathcal{D}^\star_n$, for $n \in \mathbb{Z}_{>1}$:

(i) For odd n, there is just one, since the opposite coloring of a type A bipartite graph gives an isomorphic bipartite graph in this case.

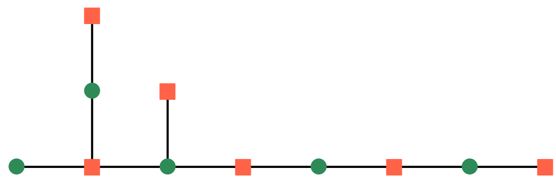
(ii) For even $n \notin \{2, 4, 12, 18, 30\}$, there are four, since the opposite colorings of A and D graphs give non-isomorphic bipartite graphs.

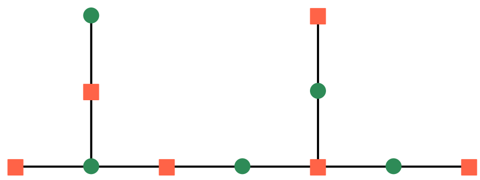
(iii) For $n \in \{2, 4\}$, there are two, since the corresponding type D graphs are of type A in these cases.

(iv) For $n \in \{12, 18, 30\}$, there are six, since there are two additional non-isomorphic bipartite graphs of type E in each case.

Example 2.20. The spectrum is a full graph invariant for type ADE graphs (see e.g. (5.1)). Thus, in order to check if two inequivalent graded simple transitive 2-representations of $\mathcal{D}_n^\star$ decategorify to isomorphic H_n-modules, we only need to compare their two-colorings. Outside types E_6 and E_8 nothing interesting happens. But the two two-colorings in Example 2.3 give inequivalent bipartite graphs which are spectrum-color-equivalent. Therefore, the corresponding graded simple transitive 2-representations of $\mathcal{D}_{12}^\star$ are inequivalent, but decategorify to isomorphic H_{12}-modules (see also Example 5.10). The same holds for the two type E_8 graded simple transitive 2-representations of $\mathcal{D}_{30}^\star$.

In the infinite case the story is more delicate. As stated above, for all bipartite graphs, i.e. not necessarily of ADE type, we can construct graded simple transitive 2-representations of $\mathcal{D}_\infty^\star$, cf. Section 4.2.5. By a classical result of Schwenk [Sch73], "almost all" trees are not determined by their spectrum – in the sense that there are other, non-isomorphic trees with the same spectrum. However, a lot of them will be spectrum-color-equivalent. Thus, already for trees there are plenty of examples of inequivalent graded simple transitive 2-representations of $\mathcal{D}_\infty^\star$ which decategorify to isomorphic H_∞-modules, e.g.

$G =$  ,

$G' =$  . ▲

3. Graded 2-representations

For us, Mazorchuk and Miemietz's setting (see e.g. [MM11], [MM16a] or [MM16b]) with finitary 2-categories and strict 2-representations is too restrictive.

The two-color Soergel calculus is defined over the graded algebra of polynomials on the geometric representation of W (*the polynomial algebra*, for short), which is finite-dimensional in each degree, but infinite-dimensional as a whole. In contrast, Mazorchuk–Miemietz always work over the so-called *coinvariant algebra*, which is a finite-dimensional quotient of the polynomial algebra. This quotient inherits a grading, but they do not use it.

Further, they consider strict 2-representations. We will define our 2-representations using the two-color Soergel calculus over the polynomial algebra and then prove that they descend to a quotient defined over the coinvariant algebra. This is done by prescribing the image of each 1-morphism and each generating 2-morphism, and checking the diagrammatic relations. On the level of 1-morphisms these 2-representations sometimes preserve composition only up to natural 2-isomorphisms, i.e. our 2-representations are given by weak 2-functors (also called non-strict or pseudo). Fortunately, every weak 2-representation in our sense can be strictified (see Remark 3.9) and e.g. the classification results from [KMMZ16] remain true in our setup.

In the following abstract setup we work over a field $\mathbb{K}$ for simplicity.

3.1. **Some basics about (graded finitary) 2-categories.**

We use strict 2-categories – which we simply call 2-categories – and bicategories. We use strict and weak 2-functors, which are carefully specified in each case.

Let $\mathbf{C}^\star$ be an additive, graded, $\mathbb{K}$-linear 2-category, i.e. a category enriched over the category of additive, graded, $\mathbb{K}$-linear categories. We always assume that $\mathbf{C}^\star$ has finitely many objects, up to equivalence, and that its 2-morphism spaces are locally finite, i.e. finite-dimensional in each degree with the grading bounded from below.

Moreover, in our setup, the 1-morphisms admit grading shifts. Let $X\{\mathtt{a}\}$ denote a given 1-morphism X shifted $\mathtt{a} \in \mathbb{Z}$ degrees such that the identity 2-morphism on X gives rise to a homogeneous 2-isomorphism $X \Rightarrow X\{\mathtt{a}\}$ of degree $\mathtt{a}$. In general, the 2-morphisms in $\mathbf{C}^{\star}$ are $\mathbb{K}$-linear combinations of homogeneous ones, where a homogeneous 2-morphism from X to Y of degree d becomes homogeneous of degree $d + \mathtt{b} - \mathtt{a}$ when seen as a 2-morphism from $X\{\mathtt{a}\}$ to $Y\{\mathtt{b}\}$.

There is a 2-subcategory $\mathbf{C}$ of $\mathbf{C}^{\star}$ which has the same objects and 1-morphisms, but only contains the degree-preserving 2-morphisms. It still has the degree shifts on 1-morphisms, but the 2-morphism spaces are no longer graded. In $\mathbf{C}$ the 1-morphisms $X\{\mathtt{a}\}$ and $X\{\mathtt{b}\}$ are in general only isomorphic for $\mathtt{a} = \mathtt{b}$.

One can recover $\mathbf{C}^{\star}$ from $\mathbf{C}$, because (for any 1-morphisms X, Y in $\mathbf{C}^{\star}$ or $\mathbf{C}$):

$$2\mathrm{Hom}_{\mathbf{C}^{\star}}(X, Y) = \bigoplus\nolimits_{\mathtt{a} \in \mathbb{Z}} 2\mathrm{Hom}_{\mathbf{C}}(X\{\mathtt{a}\}, Y).$$

We also assume that, for each pair of objects x, y, the hom-category $\mathrm{Hom}_{\mathbf{C}}(x, y)$ is idempotent complete and Krull-Schmidt. (In the diagrammatic Soergel 2-categories, we therefore have to take the Karoubi envelope of each hom-category.) We also assume that the identity 1-morphisms are indecomposable.

If the split Grothendieck group of $\mathbf{C}$ has finite rank over $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$, where v corresponds to the degree shift $\{\mathtt{1}\}$, we say that $\mathbf{C}^{\star}$ is *graded finitary*. If, additionally, the 2-morphism spaces are finite-dimensional, we say that $\mathbf{C}^{\star}$ is *graded* 2*-finitary*.

If the split Grothendieck group of $\mathbf{C}$ has countably infinite $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$-rank, we say that $\mathbf{C}^{\star}$ is *graded locally finitary*. If, additionally, the 2-morphism spaces are finite-dimensional, we say that $\mathbf{C}^{\star}$ is *graded locally* 2*-finitary*.

We also use *graded* 1*-finitary* categories, whose 1-morphism spaces are graded and finite-dimensional.

Example 3.1. The Soergel bimodules of any Coxeter type, when really defined using bimodules over the polynomial algebra R, form a bicategory. This is easy to see, e.g. $\mathrm{R} \otimes_{\mathrm{R}} \mathrm{R}$ is only isomorphic to R and not equal to it. Moreover, it is not a small 2-category, e.g. the isomorphism class of the polynomial algebra R in this 2-category is not a set. Luckily, any bicategory is weakly equivalent to a 2-category – as follows from a strictification theorem due to Mac Lane (see e.g. [ML98, Section XI.3]) for monoidal categories and to Bénabou [Bén67] for bicategories. (Alternatively, see [Lei98, Theorem 2.3].) And in our case, the 2-categories are small.

However, for our purposes we need a concrete version of such 2-categories. Thus, we use the Karoubi envelope of the two-color Soergel calculi $\mathcal{D}_n$ and $\mathcal{D}_\infty$, respectively, which we recall in Section 4.1. As we will see in Proposition 4.12, $\mathbf{Kar}(\mathcal{D}_n)^{\star}$ and $\mathbf{Kar}(\mathcal{D}_\infty)^{\star}$ are graded finitary and graded locally finitary 2-categories.

The two-color Soergel calculi are defined over the polynomial algebra, but admit quotients to 2-categories defined over the coinvariant algebra, denoted by $\mathcal{D}^{\mathrm{f}}_n$ and $\mathcal{D}^{\mathrm{f}}_\infty$. We explain this more carefully later – in the proof of part (iv) of Theorem II. By Proposition 4.12 and by construction, $\mathbf{Kar}(\mathcal{D}^{\mathrm{f}}_n)^{\star}$ and $\mathbf{Kar}(\mathcal{D}^{\mathrm{f}}_\infty)^{\star}$ are graded 2-finitary and graded locally 2-finitary 2-categories, respectively. ▲

Example 3.2. Let A be a graded, finite-dimensional algebra. Then its category of graded projective, finite-dimensional (left) A-modules A-p$\mathbf{Mod}^{\mathrm{gr}}$ is an additive, graded 1-finitary, $\mathbb{K}$-linear category which is idempotent complete and Krull-Schmidt. We view it as being small by taking equivalence classes of A-modules.

We call a finite-dimensional A-bimodule X *biprojective* if it is projective as a left and as a right A-module (but not necessarily as an A-bimodule). Given a graded biprojective, finite-dimensional A-bimodule X. Then $X \otimes_{\mathrm{A}} -$ gives rise to an exact endofunctor of A-p$\mathbf{Mod}^{\mathrm{gr}}$.

An endofunctor of A-p$\mathbf{Mod}^{\mathrm{gr}}$ is called *biprojective*, if it is isomorphic to a direct summand of a finite direct sum of endofunctors of the form $X \otimes_{\mathrm{A}} -$, where X is a biprojective A-bimodule.

Let p$\mathbf{End}$(A-p$\mathbf{Mod}^{\mathrm{gr}}$) be the 2-category with the unique object A-p$\mathbf{Mod}^{\mathrm{gr}}$, whose 1-morphisms are biprojective endofunctors on A-p$\mathbf{Mod}^{\mathrm{gr}}$, and whose 2-morphisms are degree-preserving natural transformations between these. Hereby recall that the Godement product induces the horizontal composition $\circ_h$ via:

$$\begin{gathered}\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{G}}\colon X \to Y, \qquad \boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{I}}\colon Y \to Z, \qquad \mathfrak{f}\colon \boldsymbol{\mathcal{F}} \Rightarrow \boldsymbol{\mathcal{G}}, \qquad \mathfrak{g}\colon \boldsymbol{\mathcal{H}} \Rightarrow \boldsymbol{\mathcal{I}}, \\ \mathfrak{g}\circ_h \mathfrak{f}\colon \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{F}} \Rightarrow \boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{G}}, \quad (\mathfrak{g}\circ_h \mathfrak{f})_X = \mathfrak{g}_{\mathcal{G}(X)} \circ \boldsymbol{\mathcal{H}}(\mathfrak{f}_X) = \boldsymbol{\mathcal{I}}(\mathfrak{f}_X) \circ \mathfrak{g}_{\mathcal{F}(X)}.\end{gathered} \tag{3.1}$$ ▲

Remark 3.3. All diagrammatic 2-categories which appear in this paper, e.g. $\boldsymbol{\mathcal{D}}_n$ as in Section 4.1, are, by definition, strict. In contrast, p$\mathbf{End}$(A-p$\mathbf{Mod}^{\mathrm{gr}}$), introduced in Example 3.2, is a bicategory. Still, we always view it as being strict by using the Bénabou–Mac Lane coherence theorem. ▲

3.2. **Graded simple transitive 2-representations.** Our next goal is to define a graded and weak version of certain 2-representations due to Mazorchuk–Miemietz, see e.g. [MM16a] or [MM16b] where more details can be found.

3.2.1. *2-representations: definitions.* Let $\mathfrak{A}^f_{\mathrm{gr}}$ be the 2-category whose objects are additive, graded 1-finitary, $\mathbb{K}$-linear (small) categories; 1-morphisms are additive, degree-preserving, $\mathbb{K}$-linear functors; 2-morphisms are homogeneous degree-zero natural transformations. For example, QG-p$\mathbf{Mod}^{\mathrm{gr}}$ is an object of $\mathfrak{A}^f_{\mathrm{gr}}$.

Definition 3.4. Let $\mathbf{C}^\star$ be a graded (locally) (2-)finitary 2-category. Then a *graded 2-finitary, weak 2-representation* of $\mathbf{C}^\star$ is an additive, $\mathbb{K}$-linear, weak 2-functor

$$\mathbf{M}\colon \mathbf{C}^\star \to \left(\mathfrak{A}^f_{\mathrm{gr}}\right)^\star$$

which preserves degrees and commutes with shifts as follows. For any indecomposable 1-morphism F in $\mathbf{C}^\star$ and any indecomposable object $X \in \coprod_x \mathbf{M}(x)$ (note that objects in $\coprod_x \mathbf{M}(x)$ are 1-morphisms in the target 2-category) we have

$$\mathbf{M}(F\{\mathtt{a}\})(X\{\mathtt{b}\}) = \mathbf{M}(F)(X)\{\mathtt{a}+\mathtt{b}\}, \qquad \mathtt{a},\mathtt{b} \in \mathbb{Z}.$$ ▲

These form a bicategory, whose 1-morphisms are weak natural transformations, with degree-zero structural 2-isomorphisms across the squares in their definition, and whose 2-morphisms are degree-zero modifications.

Remark 3.5. We require a graded 2-finitary, weak 2-representation to be a weak 2-functor which preserves degrees. Such a 2-functor restricts to, and is uniquely determined by, an additive, $\mathbb{K}$-linear, weak 2-functor $\mathbf{M}\colon \mathbf{C} \to \mathfrak{A}^f_{\mathrm{gr}}$. We will use both weak 2-functors almost interchangeably. ▲

Mazorchuk–Miemietz [MM16b] defined simple transitive, strict 2-representations, which are a categorical analogue of simple representations of finite-dimensional algebras. Their definition remains (almost) unchanged in the our setting.

Definition 3.6. We say that a graded 2-finitary, weak 2-representation $\mathbf{M}$ of a graded (locally) (2-)finitary 2-category $\mathbf{C}^\star$ is *transitive* if for any two indecomposable objects X, Y in $\coprod_x \mathbf{M}(x)$ there exists a 1-morphism F in $\mathbf{C}^\star$ such that Y is isomorphic to a graded direct summand of $\mathbf{M}(F)(X)$. It is called *graded simple transitive* if, additionally, $\coprod_x \mathbf{M}(x)$ has no non-zero proper $\mathbf{C}^\star$-invariant ideals. ▲

(We say "graded simple transitive" and omit the "weak", cf. Remark 3.9.)

Remark 3.7. Suppose that $Y\{\mathtt{a}\}$ is isomorphic to a direct summand of $\mathbf{M}(F)(X)$, for some $\mathtt{a} \in \mathbb{Z}$. Then Y is isomorphic to a direct summand of $\mathbf{M}(F\{-\mathtt{a}\})(X)$. ▲

Example 3.8. Let $\mathcal{D}_{\mathrm{A}_1}$ denote the one-color (say sea-green s) Soergel calculus of Coxeter type A_1. This is defined as $\mathcal{D}_n$ in Section 4.1, but dropping the second color (say tomato t) and the relations in which it is involved. This 2-category has one object (which we do not specify here). The 1-morphisms are formal direct sums of finite words ("tensor products") of shifts of s, and 2-morphisms are degree zero Soergel diagrams. The 1-morphisms might not be indecomposable, e.g. we have $ss \cong s\{-1\} \oplus s\{+1\}$. In particular, $\mathcal{D}_{\mathrm{A}_1}$ is an additive, $\mathbb{K}$-linear 2-category and $\mathcal{D}_{\mathrm{A}_1} \cong \mathbf{Kar}(\mathcal{D}_{\mathrm{A}_1})$ is idempotent complete and Krull-Schmidt. Finally, $\mathbf{Kar}(\mathcal{D}_{\mathrm{A}_1})^{\star}$ is a graded finitary 2-category.

Now, consider the coinvariant algebra $\mathrm{C}^{+}_{\mathrm{A}_1} = \mathrm{D}$ of the Weyl group of type A_1, the so-called dual numbers $\mathrm{D} \cong \mathbb{K}[X]/(X^2)$ – with X of degree two. Then the 2-category $\mathcal{D}^{\mathrm{f}}_{\mathrm{A}_1}$ which is defined over the coinvariant algebra, is a quotient of $\mathcal{D}_{\mathrm{A}_1}$. The $\star$ of its Karoubi envelope $\mathbf{Kar}(\mathcal{D}^{\mathrm{f}}_{\mathrm{A}_1})^{\star}$ is a graded 2-finitary 2-category.

We can define a graded 2-finitary, weak 2-representation of $\mathbf{Kar}(\mathcal{D}_{\mathrm{A}_1})^{\star}$ on the category $(\mathrm{D}\text{-p}\mathbf{Mod}^{\mathrm{gr}})^{\star}$, by sending the empty word $\emptyset$ to the endofunctor $\mathrm{D} \otimes_{\mathrm{D}} -$ and s to the endofunctor $\mathrm{D} \otimes \mathrm{D} \otimes_{\mathrm{D}} -$. Note that D and $\mathrm{D} \otimes \mathrm{D}$ are graded biprojective, D-bimodules. This 2-representation is simple transitive, see [MM17, Section 3.4], and also graded. By construction, it descends to $\mathbf{Kar}(\mathcal{D}^{\mathrm{f}}_{\mathrm{A}_1})^{\star}$.

In the setup of bipartite graphs, this 2-representation is given by a graph with one sea-green s colored vertex $\bullet$. One can check that there is no other graded simple transitive 2-representation for the corresponding small quotient. ▲

3.2.2. *2-representations: strictifications.* Let $\mathbf{C}$ be a small 2-category, $\mathbf{Cat}$ the 2-category of small categories and $\mathcal{F}_{\mathrm{w}} \colon \mathbf{C} \to \mathbf{Cat}$ any weak 2-functor. By [Pow89, Section 4.2] this 2-functor can be strictified: There exists a strict 2-functor $\mathcal{F}_{\mathrm{s}} \colon \mathbf{C} \to \mathbf{Cat}$ and a weak natural 2-isomorphism $\mathfrak{f} \colon \mathcal{F}_{\mathrm{w}} \Rightarrow \mathcal{F}_{\mathrm{s}}$. Moreover, given two such weak 2-functors $\mathcal{F}_{\mathrm{w}}$ and $\mathcal{G}_{\mathrm{w}}$, the 2-categories of weak – respectively strict – natural transformations and modifications between $\mathcal{F}_{\mathrm{w}}$ and $\mathcal{G}_{\mathrm{w}}$, respectively between $\mathcal{F}_{\mathrm{s}}$ and $\mathcal{G}_{\mathrm{s}}$, are equivalent by conjugation with $\mathfrak{f}$ and $\mathfrak{g}$.

A close inspection of Power's arguments shows that the same holds when $\mathbf{Cat}$ is replaced by $\mathfrak{A}^{f}_{\mathrm{gr}}$, as long as $\mathbf{C}$ is graded (locally) (2-)finitary with finitely many objects.

(We thank Nick Gurski for explaining to us the relevance of 2-monads for strictification and giving us some pointers to the literature.)

Remark 3.9. Note that the two-color Soergel calculi (and their Karoubi envelopes), in the finite and the infinite case, are 2-categories with finitely many objects. Thus, by the above, the (graded simple transitive) weak 2-representations in this paper are weakly equivalent to (graded simple transitive) strict 2-representations. ▲

4. The two-color Soergel calculus and its 2-action

Next, we construct the 2-representations from Theorem I.

4.1. **Soergel calculus in two colors.** Fix $n \in \mathbb{Z}_{>0}$ or "$n = \infty$". We recall Elias' two-color Soergel calculi $\mathcal{D}^{\star}_n$ and $\mathcal{D}^{\star}_{\infty}$, as defined in [Eli16, Sections 5 and 6]. We write $\mathcal{D}^{\star}$ if no confusion can arise, and also use "unstarred" versions $\mathcal{D}$ of these.

Remark 4.1. Let us now fix the technical conditions defining our ground rings.

For weak categorifications, as in Section 2.3.4, we can work over $\mathscr{A}_{\mathrm{gen}} = \mathbb{Z}[\mathrm{q}, \mathrm{q}^{-1}]$ for a fixed (generic) $\mathrm{q} \in \mathbb{C} - \{0\}$.

For strong categorifications, as in Section 2.4, we have to adjoin the inverse of some quantum integers as well. For G of ADE type with Coxeter number $n > 2$ (cf. (A), (D) or (E)), let q be a complex, primitive $2n$th root of unity and let

$$\mathscr{A}_n = \mathbb{Z}[1/2, \mathrm{q}, \mathrm{q}^{-1}, 1/[2]_{\mathrm{q}}, \dots, 1/[n{-}1]_{\mathrm{q}}]. \tag{4.1}$$

For $n = 1$ and $n = 2$ we let $\mathscr{A}_n = \mathbb{Z}[1/2]$ and $\mathbb{Z}[1/2, \pm\sqrt{-1}]$, respectively. Hereby recall that the quantum integers are defined by

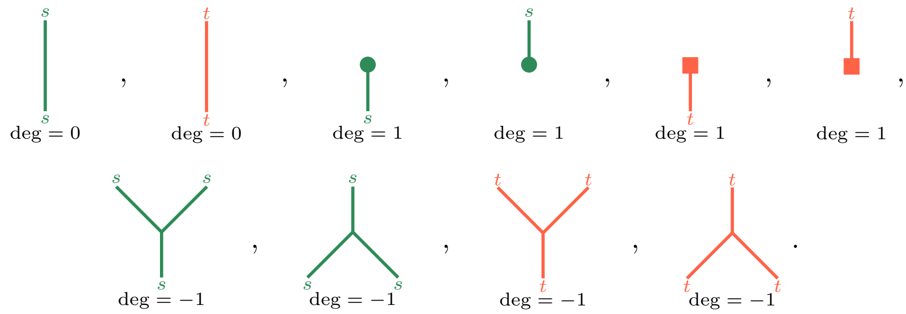
$$[z]_{\mathrm{q}} = \mathrm{q}^{z-1} + \mathrm{q}^{z-3} + \cdots + \mathrm{q}^{3-z} + \mathrm{q}^{1-z}, \ z \in \mathbb{Z}_{>0}, \quad [0]_{\mathrm{q}} = 0.$$

For more general bipartite graphs we use $\mathscr{A}_{G,\vec{\lambda},\mathrm{q}}$ as in Section 4.2.5, which is obtained from $\mathbb{Z}$ by adjoining a finite number of complex numbers and their inverses. To be precise, we adjoin the (4.BF2)-weightings as in Section 4.2.3. ▲

The definition of those calculi depends on a fixed parameter $\mathrm{q} \in \mathbb{C} - \{0\}$, which is as in Section 2.3 and Remark 4.1. Our ground ring is again the appropriate $\mathscr{A}$.

4.1.1. *Soergel diagrams.* We consider the following generating 2-morphisms.

(4.$\boldsymbol{\mathcal{D}}$gen1)

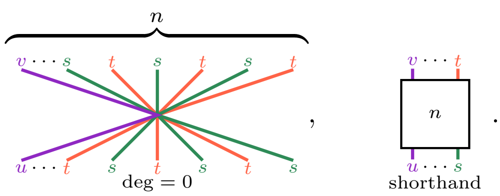


(Recall that the objects s and t correspond to θ_s and θ_t of H, and not to the Coxeter generators s and t of W, see e.g. [Eli16, Theorems 5.29 and 6.24].)

We give these 2-morphisms the indicated degrees, and call them *identities*, (*end* and *start*) *dots*, and *trivalent vertices* (*split* and *merge*) respectively.

All generators in (4.$\boldsymbol{\mathcal{D}}$gen1) are independent of n. The following degree-zero generator depends on n and is called the $2n$-*vertex*:

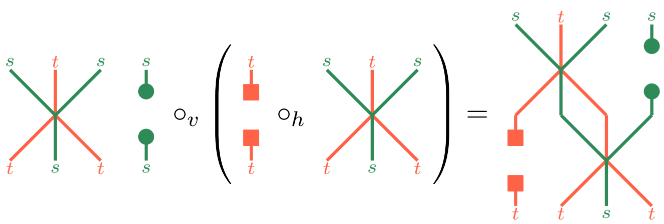
(4.$\boldsymbol{\mathcal{D}}$gen2)



Here u is either s or t, and v is the opposite, depending on n. The color inverted ($s \rightleftarrows t$) version of the $2n$-vertex in (4.$\boldsymbol{\mathcal{D}}$gen2) is our last generator. Throughout the paper, we will use the shorthand for the $2n$-vertices as in (4.$\boldsymbol{\mathcal{D}}$gen2).

The vertical composition $g \circ_v f$ is given by gluing diagram g on top of diagram f (in case the colors match), the horizontal composition $g \circ_h f$ by putting g to the left of f. Our conventions are best illustrated in an example.

Example 4.2. For instance, in case $n = 3$:



: $tstt \to ssts$.

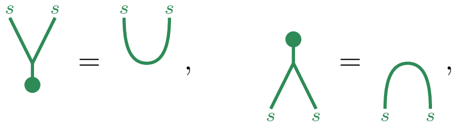
Here we have also indicated our reading conventions for Soergel diagrams. ▲

Definition 4.3. Given two words w, w' in the symbols s and t, a Soergel diagram from w to w' is a diagram $\circ_v$-$\circ_h$-generated by the generators from (4.$\mathcal{D}$gen1) and (4.$\mathcal{D}$gen2) such that the outgoing edges correspond color-wise to the entries of w, w'.

The degree of a Soergel diagram is, by definition, the sum of the degrees of its generators. (The Soergel diagram $\varnothing\colon \emptyset \to \emptyset$ is, by convention, of degree zero.) ▲

The top and bottom of a Soergel diagram correspond to direct sums (we do not assume that words in s and t are reduced) of so-called *Bott–Samelson bimodules*, which categorify Bott–Samelson basis elements $\theta_{\overline{w}}$, for $w \in \mathrm{W}$, cf. Section 2.1.2. In general, the Bott–Samelson bimodules do not coincide with the indecomposable Soergel bimodules, which categorify the Kazhdan–Lusztig basis elements θ_w.

Example 4.4. The following shorthand notations and their color inversion ($s \rightleftarrows t$)

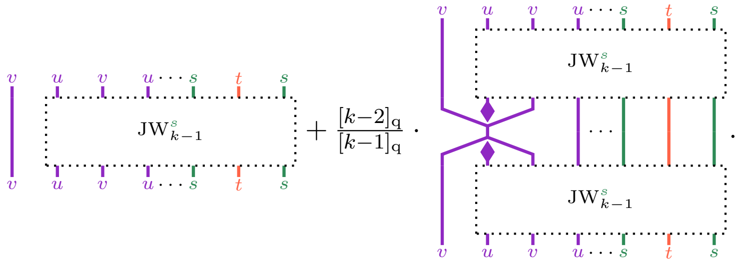


define cup and cap Soergel diagrams, which are of degree zero. ▲

4.1.2. *"Dihedral Jones–Wenzl projectors".* From now on we write id for Soergel diagrams consisting of just identity strands.

Definition 4.5. For $k \in \mathbb{Z}_{\geq 0}$, we define JW^s_k to be the formal $\mathscr{A}$-linear combination of Soergel diagram obtained as follows. Set $\mathrm{JW}^s_0 = \varnothing$, $\mathrm{JW}^s_1 = \mathrm{id}$ and

(4.4) $\mathrm{JW}^s_k =$

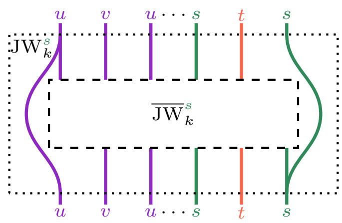


Similarly, we define JW^t_k. Note that for any $k \in \mathbb{Z}_{\geq 0}$, JW^s_k is a $\mathscr{A}$-linear combination of degree zero Soergel diagrams. ▲

If q is not a root of unity or q is equal to ± 1, then JW^u_k is well-defined for any $k \in \mathbb{Z}_{\geq 0}$ (if one works in e.g. $\mathbb{C}$). If q is a complex, primitive $2n$th root of unity, then JW^u_k (with recursion as in Definition 4.5) is well-defined for $0 \leq k \leq n$, and we can work over $\mathscr{A}_n$. Moreover, we require q to be a complex, primitive $2n$th root of unity in the proof of Proposition 4.12 in order to make JW^u_{n-1} rotationally invariant, see e.g. [Eli16, before Proposition 1.2].

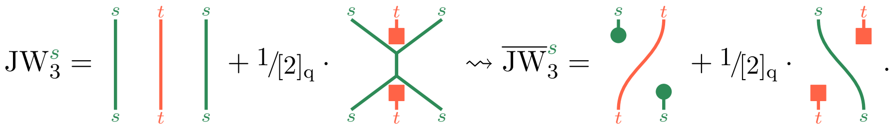
Remark 4.6. Formula (4.4) comes from Wenzl's formula for the Jones–Wenzl projectors in the Temperley–Lieb algebra, see [Wen87]. See also [AT17, Definition 4.12]. ▲

Remark 4.7. From JW^u_k we obtain a certain diagram $\overline{\mathrm{JW}}^u_k$. Since we do not need it very often in this paper, we refer to [Eli16, below Definition 5.14] for its definition. For our purposes it is enough to know that any 2-functor maps $\overline{\mathrm{JW}}^u_k$ to zero if and only if it maps JW^u_k to zero. The picture to keep in mind is:



using the convention that "open" strings are to be closed with dots. This is almost a definition of $\overline{\mathrm{JW}}^{s}_{k}$, given the relations in the Soergel calculus. ▲

Example 4.8. With three strands we have the following:



More examples can be found in e.g. [Eli16, Examples 5.16 and 5.17]. ▲

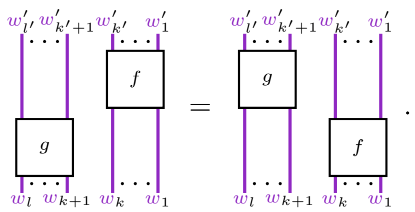
4.1.3. *The two-color Soergel calculus.* Recall that we have fixed $n \in \mathbb{Z}_{>0}$ or "$n = \infty$". In the first case, let q be any primitive $2n$th root of unity. In the second case, let q be any non-zero complex number.

Definition 4.9. Denote by $\mathcal{D}_n$ the additive closure of the $\mathscr{A}$-linear 2-category, determined by the following data:

(i) There is one (not further specified) object.
(ii) The 1-morphisms are formal shifts of finite words w in the symbols s and t. (To simplify notation, we often omit these shifts in the diagrams.)
(iii) The 2-morphism space $2\mathrm{Hom}_{\mathcal{D}_n}(w\{\mathtt{a}\}, w'\{\mathtt{b}\})$ is the $\mathscr{A}$-linear span of all Soergel diagrams from w to w' of degree $\mathtt{b} - \mathtt{a}$, quotiented by the relations from (4.EH) to (4.2nv3).
(iv) Vertical composition $\circ_v$ is induced by the vertical gluing of Soergel diagrams, while the horizontal composition $\circ_h$ is induced by putting diagrams next to each other horizontally. (Using the same reading conventions as above.)

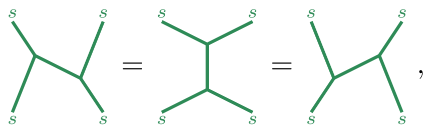
For the usual Eckmann–Hilton relation for 2-morphisms to hold, we have to impose the far-commutativity relation (here f, g are two arbitrary Soergel diagrams):

(4.EH)

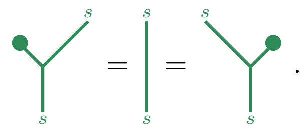


The other relations among Soergel diagrams are the following. First, the Frobenius relations (including the horizontal mirror of (4.Fr2)):

(4.Fr1)

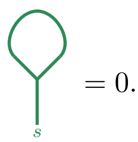


(4.Fr2)

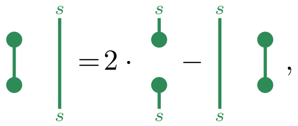
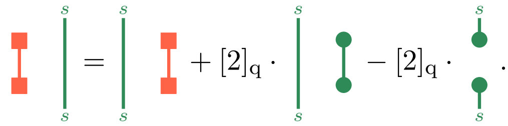


Then the needle relation (including its horizontal mirror):

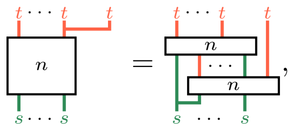
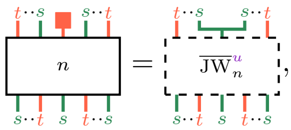
(4.Ne)

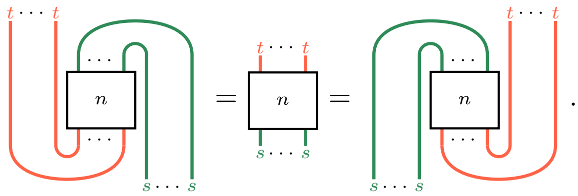


The next relations, still independent of n, are the barbell forcing relations:

(4.BF1)  (4.BF2') 

Finally, three relations which depend on n (including the horizontal mirror of the displayed ones, the versions for even n, and all possibilities for the position of the dot in (4.2nv2)):

(4.2nv1)  (4.2nv2) 

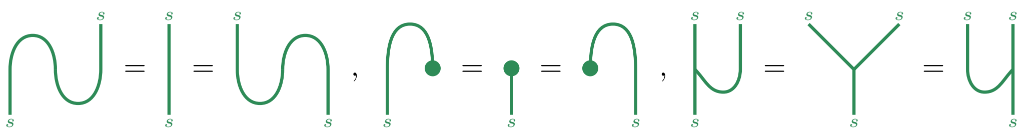
(4.2nv3) 

Moreover, all of the relations listed above, one- and two-colored, exist in two versions, i.e. the displayed ones and their color inverted ($s \rightleftarrows t$) counterparts. ▲

The 2-category $\mathcal{D}_\infty$ is defined similarly, but using only the generators and relations which are independent of n.

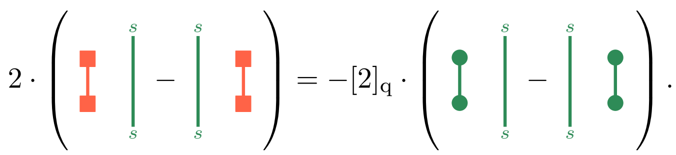
Let $\mathcal{D}^\star$ denote the 2-category obtained from $\mathcal{D}$ as explained in Section 3.1. Since the relations (4.EH) to (4.2nv2) are homogeneous with respect to the degrees of the Soergel diagrams, the 2-category $\mathcal{D}^\star$ is additive, graded and $\mathscr{A}$-linear.

Example 4.10. Isotopy relations such as "zigzag relations" and other as e.g.



are consequences of the Frobenius relations from (4.Fr1) and (4.Fr2). However, the isotopy relations in (4.2nv3) are not consequences of the Frobenius relations. ▲

Remark 4.11. When 2 is invertible it is not hard to show that (4.BF2') can actually be replaced by

(4.BF2) 

We use this equivalent relation later in Section 4.2.3. ▲

The following is a direct consequence of [Eli16, Theorems 5.29 and 6.24]. To be consistent with our conventions from Section 3, we switch to a field $\mathbb{K}$ containing $\mathscr{A}$.

Proposition 4.12. The 2-categories $\mathbf{Kar}(\boldsymbol{\mathcal{D}}_n)^{\star}$ and $\mathbf{Kar}(\boldsymbol{\mathcal{D}}_{\infty})^{\star}$ are graded finitary and graded locally finitary, respectively. ■

4.2. **Its 2-representations coming from bipartite graphs.** The purpose of the present subsection is to define the weak 2-functors

$$\boldsymbol{\mathcal{G}}\colon \boldsymbol{\mathcal{D}}^{\star} \to \mathrm{p}\mathbf{End}^{\star}(\mathbf{G}^{\mathrm{gr}})$$

(in our usual convention, we write $\boldsymbol{\mathcal{G}}$ for either $\boldsymbol{\mathcal{G}}_{\infty}$ or $\boldsymbol{\mathcal{G}}_n$) from Theorem I, for any given bipartite graph G. As we will see in Section 5.2, the 2-functor $\boldsymbol{\mathcal{G}}$ is well-defined only for certain values of $\mathrm{q} \in \mathbb{C} - \{0\}$, which depend on G.

Now, $\boldsymbol{\mathcal{G}}$ sends the unique object of $\boldsymbol{\mathcal{D}}^{\star}$ to $\mathbf{G}^{\mathrm{gr}}$. In Section 2.3 we already defined $\boldsymbol{\mathcal{G}}$ on 1-morphisms:

$$\boldsymbol{\mathcal{G}}(s) = \boldsymbol{\Theta}_s, \quad \boldsymbol{\mathcal{G}}(t) = \boldsymbol{\Theta}_t.$$

Next, we define $\boldsymbol{\mathcal{G}}$ on 2-morphisms, which we first do for $\boldsymbol{\mathcal{D}}^{\star}_{\infty}$ and then for $\boldsymbol{\mathcal{D}}^{\star}_{n}$.

Moreover, we start with G's of ADE type (our main interest), and then discuss the generalization to arbitrary bipartite graphs.

4.2.1. *Assignment in the infinite case.* Fix $\mathrm{q} \in \mathbb{C} - \{0\}$. We have to assign natural transformations to the generators from (4.$\boldsymbol{\mathcal{D}}$gen1). Recall that the functors $\boldsymbol{\Theta}_s$ and $\boldsymbol{\Theta}_t$ are given by tensoring with the sum of the $\mathrm{Q}G$-bimodules $P_{\mathrm{i}}\{-1\} \otimes {}_{\mathrm{i}}P$ over all sea-green $\underline{\mathrm{i}}$ and over all tomato $\overline{\mathrm{j}}$ colored vertices of G, respectively. Thus, $\mathrm{Q}G$-bimodule maps between tensor products of the $\mathrm{Q}G$-bimodules $P_{\mathrm{i}}\{-1\} \otimes {}_{\mathrm{i}}P$ induce natural transformations between the corresponding composites of $\boldsymbol{\Theta}_s$ and $\boldsymbol{\Theta}_t$. (This assignment is not strict, cf. Example 3.1.) Hence, we first specify a $\mathrm{Q}G$-bimodule map for each of the generating Soergel diagrams and then check that our assignment preserves the relations (4.EH) to (4.BF2) of the two-color Soergel calculus.

To understand our assignment below recall that, for all vertices $\mathrm{i} \in G$, there are free $\mathscr{A}$-modules ${}_{\mathrm{i}}P_{\mathrm{i}}$ given by

$$ {}_{\mathrm{i}}P_{\mathrm{i}} = {}_{\mathrm{i}}P \,\widehat{\otimes}\, P_{\mathrm{i}} \cong \mathscr{A}(\mathrm{i}) \oplus \mathscr{A}(\mathrm{i}|\mathrm{i}), \tag{4.15}$$

where the isomorphism, which we fix, is given by $y \,\widehat{\otimes}\, x \mapsto y \bullet x$. We will use (4.15) strategically below. Moreover, there are graded $\mathrm{Q}G$-bimodules

$$\begin{aligned} P_{\mathrm{i}}\{-1\} \otimes {}_{\mathrm{i}}P \cong &\left(\mathscr{A}(\mathrm{i}) \oplus \mathscr{A}(\mathrm{i}|\mathrm{i}) \oplus \textstyle\bigoplus_{\mathrm{i}-\mathrm{j}} \mathscr{A}(\mathrm{j}|\mathrm{i})\right)\{-1\} \\ &\otimes \left(\mathscr{A}(\mathrm{i}) \oplus \mathscr{A}(\mathrm{i}|\mathrm{i}) \oplus \textstyle\bigoplus_{\mathrm{i}-\mathrm{j}} \mathscr{A}(\mathrm{i}|\mathrm{j})\right), \end{aligned}$$

$${}_{\mathrm{j}}P_{\mathrm{i}} = {}_{\mathrm{j}}P \,\widehat{\otimes}\, P_{\mathrm{i}} \cong \mathscr{A}(\mathrm{j}|\mathrm{i}),$$

which easily follows from (2.10) and the same isomorphism as before, respectively. Hereby we recall that the left action is given by post-composition, and the right action by pre-composition of paths.

In the following we only give some of the $\mathrm{Q}G$-bimodule maps. The other ones can be obtained from these via color inversion ($s \rightleftarrows t$) and interchanging $\underline{\mathrm{S}}$ and $\overline{\mathrm{T}}$.

Some of the maps below are weighted sums. The *weights* λ_{i} are invertible elements of $\mathscr{A}$, which depend on G, and will be defined in Definition 4.16.

In the assignment below, we fix $\underline{\mathrm{i}} \in G$, while $\overline{\mathrm{j}}$ always means a vertex of G connected to $\underline{\mathrm{i}}$. We then give each $\mathrm{Q}G$-bimodule map only on certain basis elements. The rest of the map is determined by the fact that it is supposed to be a $\mathrm{Q}G$-bimodule map. (We will check in Lemma 5.6 that our assignments are indeed $\mathrm{Q}G$-bimodule maps.)

In order to get started, let $x_{\underline{\mathrm{i}}} \in P_{\underline{\mathrm{i}}}$ and ${}_{\underline{\mathrm{i}}}y \in {}_{\underline{\mathrm{i}}}P$, and let us write $\bigoplus_{\underline{\mathrm{i}}} = \bigoplus_{\underline{\mathrm{i}} \in G}$ etc. for short, and notations of the form $x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y$ indicate that we take the corresponding entries from the direct sums. (We extend the below $\mathscr{A}$-linearly.)

Identity generators. To these we assign the corresponding identity maps.

Dots. We choose the following $\mathrm{Q}G$-bimodule maps.

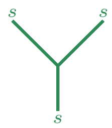
(4.d1)

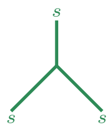
$$\overset{\mathcal{G}}{\rightsquigarrow} \begin{cases} \bigoplus_{\underline{\mathrm{i}}} P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P & \to \mathrm{Q}G, \\ x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y & \mapsto x_{\underline{\mathrm{i}}} \bullet {}_{\underline{\mathrm{i}}}y, \end{cases}$$

(4.d2)

$$\overset{\mathcal{G}}{\rightsquigarrow} \begin{cases} \mathrm{Q}G & \to \bigoplus_{\underline{\mathrm{i}}} P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P, \\ \underline{\mathrm{i}} & \mapsto \lambda_{\underline{\mathrm{i}}} \cdot (\underline{\mathrm{i}} \otimes \underline{\mathrm{i}}|\underline{\mathrm{i}} + \underline{\mathrm{i}}|\underline{\mathrm{i}} \otimes \underline{\mathrm{i}}), \\ \underline{\mathrm{j}} & \mapsto \sum_{\underline{\mathrm{i}} - \underline{\mathrm{j}}} (\lambda_{\underline{\mathrm{i}}} \cdot \underline{\mathrm{j}}|\underline{\mathrm{i}} \otimes \underline{\mathrm{i}}|\underline{\mathrm{j}}). \end{cases}$$

Trivalent vertices. Using the identification from (4.15), we pick:

(4.t1)

$$\overset{\mathcal{G}}{\rightsquigarrow} \begin{cases} \bigoplus_{\underline{\mathrm{i}}} P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P & \to \bigoplus_{\underline{\mathrm{i}}} P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P, \\ x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y & \mapsto x_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}y, \end{cases}$$

(4.t2)

$$\overset{\mathcal{G}}{\rightsquigarrow} \begin{cases} \bigoplus_{\underline{\mathrm{i}}} P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P & \to \bigoplus_{\underline{\mathrm{i}}} P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P, \\ x_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}y & \mapsto 0, \\ x_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}}|\underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}y & \mapsto \lambda_{\underline{\mathrm{i}}}^{-1} \cdot (x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y). \end{cases}$$

Tomato t colored diagrams. We inverted the colors ($s \rightleftarrows t$), including $\lambda_{\underline{\mathrm{i}}} \rightleftarrows \lambda_{\underline{\mathrm{j}}}$.

Our assignments above extend to any 2-morphism in $\boldsymbol{\mathcal{D}}_{\infty}^{\star}$ which is written as a horizontal and vertical composite of the generators (using (3.1)).

Example 4.13. Let us denote by $\mathfrak{d}$ the natural transformation induced by the $\mathrm{Q}G$-bimodule map from (4.d1). Consider the following two natural transformations.


$$\overset{\mathcal{G}}{\rightsquigarrow} \mathfrak{i}\mathfrak{d} \circ_h \mathfrak{d} \colon \boldsymbol{\Theta}_s \boldsymbol{\Theta}_s \Rightarrow \boldsymbol{\Theta}_s,$$

$$\overset{\mathcal{G}}{\rightsquigarrow} \mathfrak{d} \circ_h \mathfrak{i}\mathfrak{d} \colon \boldsymbol{\Theta}_s \boldsymbol{\Theta}_s \Rightarrow \boldsymbol{\Theta}_s.$$

Now – by the above – we have using (4.15) (which we will do silently from now on)

$$\mathfrak{i}\mathfrak{d} \circ_h \mathfrak{d} \rightsquigarrow \begin{cases} \bigoplus_{\underline{\mathrm{i}}} P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P & \to \bigoplus_{\underline{\mathrm{i}}} P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P, \\ x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y' \bullet x'_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y & \mapsto x_{\underline{\mathrm{i}}} \otimes \left({}_{\underline{\mathrm{i}}}y' \bullet x'_{\underline{\mathrm{i}}}\right) \bullet {}_{\underline{\mathrm{i}}}y. \end{cases}$$

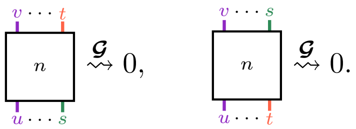
$$\mathfrak{d} \circ_h \mathfrak{i}\mathfrak{d} \rightsquigarrow \begin{cases} \bigoplus_{\underline{\mathrm{i}}} P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P & \to \bigoplus_{\underline{\mathrm{i}}} P_{\underline{\mathrm{i}}}\{-1\} \otimes {}_{\underline{\mathrm{i}}}P, \\ x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y' \bullet x'_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y & \mapsto x_{\underline{\mathrm{i}}} \bullet \left({}_{\underline{\mathrm{i}}}y' \bullet x'_{\underline{\mathrm{i}}}\right) \otimes {}_{\underline{\mathrm{i}}}y. \end{cases}$$

Hereby $x_{\underline{\mathrm{i}}}, x'_{\underline{\mathrm{i}}} \in P_{\underline{\mathrm{i}}}$ and ${}_{\underline{\mathrm{i}}}y, {}_{\underline{\mathrm{i}}}y' \in {}_{\underline{\mathrm{i}}}P$, as before. ▲

4.2.2. *Assignment in the finite case.* Let G be of ADE type and q be – as usual – a complex, primitive $2n$th root of unity. We continue to write $\boldsymbol{\mathcal{G}}$, whose definition we need to complete on the $2n$-valent vertices. To this end, recall $\boldsymbol{\Theta}_{\overline{s_n}}$ and $\boldsymbol{\Theta}_{\overline{t_n}}$ from Section 2.1.

$2n$*-valent vertices.* We assign the zero maps:

(4.2nv)



These give rise to the zero natural transformations between $\boldsymbol{\Theta}_{\overline{s_n}}$ and $\boldsymbol{\Theta}_{\overline{t_n}}$, and between $\boldsymbol{\Theta}_{\overline{t_n}}$ and $\boldsymbol{\Theta}_{\overline{s_n}}$, respectively Again, we extend everything horizontally (using (3.1)).

4.2.3. *The weighting.* Fix any $\mathrm{q} \in \mathbb{C} - \{0\}$. Recall that $V = \underline{\mathrm{S}} \coprod \overline{\mathrm{T}}$ denotes the two-colored edge set of G.

Definition 4.14. A *weighting* $\vec{\lambda}$ *of* G is an assignment

$$V \to \mathbb{C} - \{0\}, \quad \underline{\mathrm{i}} \mapsto \lambda_{\underline{\mathrm{i}}},\ \bar{\mathrm{j}} \mapsto \lambda_{\bar{\mathrm{j}}}.$$

The scalars λ_{i} are called *weights*.

Fixing an ordering of the vertices of G as in Section 2.2.2 we can write

$$\vec{\lambda} = (\lambda_{\underline{\mathrm{i}}_1}, \dots, \lambda_{\underline{\mathrm{i}}_{|\underline{\mathrm{S}}|}}, \lambda_{\bar{\mathrm{j}}_1}, \dots, \lambda_{\bar{\mathrm{j}}_{|\overline{\mathrm{T}}|}}).$$

Then the triple $(G, \vec{\lambda}, \mathrm{q})$ is called a (4.BF2)-*weighting* if

$$A(G)\vec{\lambda} = -[2]_{\mathrm{q}} \cdot \vec{\lambda}, \tag{4.21}$$

where $A(G)$ is the adjacency matrix of G (in the evident ordering). ▲

Note that (4.21) is equivalent to

$$-[2]_{\mathrm{q}} \cdot \lambda_{\mathrm{i}} = \textstyle\sum_{\mathrm{i} - \mathrm{j}} \lambda_{\mathrm{j}}, \text{ for all } \mathrm{i} \in G. \tag{4.22}$$

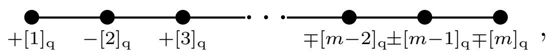
Remark 4.15. Let us explain where the condition (4.22) comes from. Recall that we assume that 2 is invertible, and we can replace (4.BF2') by (4.BF2), cf. Remark 4.11.

Thus, in order for the corresponding sums of (5.2) and (5.5) to work out (these appear in the proof that (4.BF2) holds in the 2-representation), we need precisely condition (4.22) to be satisfied. Hence, the name (4.BF2)-weighting. ▲

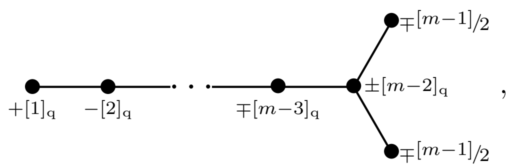
Definition 4.16. Let G be of ADE type and n be its Coxeter number, where we use the conventions from (A), (D) and (E). Let q be the usual (fixed) complex, primitive $2n$-root of unity.

We define (4.BF2)-weightings for these G's as follows. The scalars λ_{i} do not depend on the two-coloring, and are given next to the vertices.

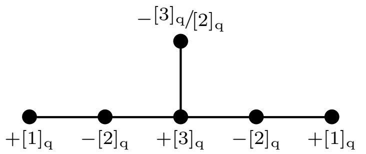
(A_m)



(D_m)

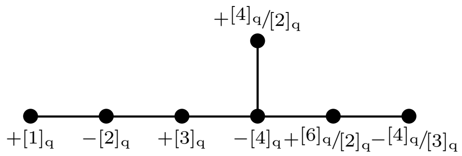


(E_6)



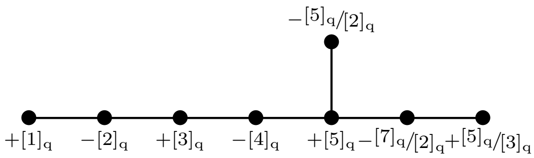
,

(E_7)



,

(E_8)



.

This defines the triple $(G, \vec{\lambda}, \mathrm{q})$. ▲

Remark 4.17. These λ_{i}'s can be found solving the eigenvector problem for the adjacency matrix of G, cf. (4.21). The weights are actually the entries of the Perron–Frobenius eigenvector – normalized to have the entry 1 at a fixed vertex – for the smallest eigenvalue. (Recall hereby that the spectrum of a bipartite graph is a symmetric set. Hence, there is a unique smallest eigenvalue. See also in the proof of Lemma 4.21.)

The reader familiar with [KJO02] might also note that the weights above are equal to the quantum numbers associated to the vertices of G in [KJO02, Section 6], after division by the quantum number associated to the left-most vertex (and adding signs to match our conventions). ▲

It remains to prove well-definedness, which we will do in Section 5, i.e. we will show that the 2-functor $\mathcal{G}$ is well-defined if and only if the chosen weighting is (4.BF2).

4.2.4. *"Uniqueness" of the* 2*-functor* $\mathcal{G}$*.* For G of ADE type, the assignment from above is essentially unique:

Lemma 4.18. Let G be of ADE type and fix q to be a complex, primitive $2n$th root of unity. For any additive, degree-preserving, $\mathbb{C}$-linear, weak 2-functor

$$\mathcal{H}\colon \mathcal{D}_{\infty}^{\star} \to \mathrm{p}\mathbf{End}^{\star}(\mathbf{G}^{\mathrm{gr}})$$

which agrees with $\mathcal{G}$ on 1-morphisms, there exist scalars $\tau, \upsilon \in \mathbb{C} - \{0\}$ such that

$$\mathcal{H}(f) = \tau^{d_{\mathrm{st}} - t_{\mathrm{me}}} \cdot \upsilon^{d_{\mathrm{end}} - t_{\mathrm{sp}}} \cdot \mathcal{G}(f),$$

for any homogeneous 2-morphism f built from d_{st} start dots, t_{me} merges, d_{end} end dots and t_{sp} splits.

In particular, $\mathcal{H}$ is equivalent to $\mathcal{G}$. □

Lemma 4.19. Any additive, degree-preserving, $\mathbb{C}$-linear, weak 2-functor

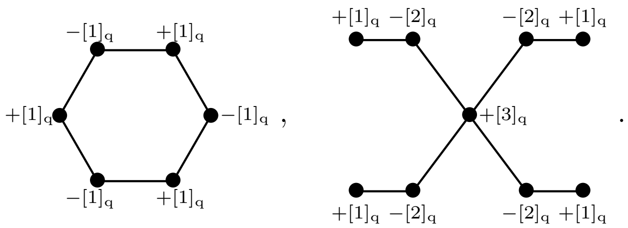
$$\mathcal{H}\colon \mathcal{D}_{n}^{\star} \to \mathrm{p}\mathbf{End}^{\star}(\mathbf{G}^{\mathrm{gr}})$$

which agrees with $\mathcal{G}$ on 1-morphisms sends the $2n$-valent vertices to zero. □

4.2.5. *More general bipartite graphs.* We will now briefly discuss (4.BF2)-weightings for graphs which are not of ADE type.

We first give two examples:

Example 4.20. Here are two examples for G's being not of ADE type:



The first weighting is (4.BF2) if and only if $-[2]_q \cdot \pm[1]_q = 2 \cdot \mp[1]_q$, i.e. if and only if $q = 1$. The second weighting is (4.BF2) if and only if $-[4]_q + 3 \cdot [2]_q = 0$ and all involved weights are non-zero. This happens if and only if $q \in \{1/2 \cdot (1 \pm \sqrt{5}), -1/2 \cdot (1 \pm \sqrt{5})\}$. ▲

In general, we will prove the following result.

Lemma 4.21. For any bipartite graph G, there is at least one value of $q \in \mathbb{C} - \{0\}$ such that a (4.BF2)-weighting exists. □

Note that Lemma 4.21 is not immediate, since we need all weights to be invertible.

5. Proofs

Finally, we give the proofs of all statements.

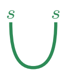
5.1. **The uncategorified story.** Fix a bipartite graph G. The following lemma follows from the fact that the $P_i\{-1\} \otimes {}_iP$'s are graded biprojective QG-bimodules.

Lemma 5.1. The functors $\mathbf{\Theta}_s$ and $\mathbf{\Theta}_t$ are degree-preserving and biprojective. ■

Lemma 5.2. The functors $\mathbf{\Theta}_s$ and $\mathbf{\Theta}_t$ are additive, $\mathscr{A}$-linear and self-adjoint. □

Proof. We only treat the case of $\mathbf{\Theta}_s$ here, the other is similar and omitted.

Up to the self-adjointness of $\mathbf{\Theta}_s$ the statement is clear. Moreover, we claim that



$$\overset{\mathcal{G}}{\rightsquigarrow} \mathfrak{i} \colon \mathcal{ID} \Rightarrow \mathbf{\Theta}_s\mathbf{\Theta}_s, \qquad$$

$$\overset{\mathcal{G}}{\rightsquigarrow} \mathfrak{e} \colon \mathbf{\Theta}_s\mathbf{\Theta}_s \Rightarrow \mathcal{ID},$$

form the unit respectively counit of the self-adjunction for $\mathbf{\Theta}_s$. Next, there are two things to check. Namely that $\mathbf{\Theta}_s$ is left adjoint to itself, and that it is right adjoint to itself, i.e. we have to show that

$$\mathfrak{id}_{\mathbf{\Theta}_s} = \mathfrak{e}\mathbf{\Theta}_s \circ_v \mathbf{\Theta}_s\mathfrak{i}, \qquad \mathfrak{id}_{\mathbf{\Theta}_s} = \mathbf{\Theta}_s\mathfrak{e} \circ_v \mathfrak{i}\mathbf{\Theta}_s.$$

Recalling the definitions of cup and cap from Example 4.4, we see that graphically this is the "zigzag relation" from Example 4.10. Hence, the claim of self-adjointness follows from the well-definedness of $\mathcal{G}$, which we show later in Section 5.2. ■

Alternatively, after checking that the underlying quiver algebra is weakly symmetric and self-injective – which can be done by e.g. copying [HK01, Proposition 1] – one could also use [MM11, Section 7.3] to prove Lemma 5.2 abstractly. In contrast, our proof above fixes the natural transformations realizing the adjunctions.

Proof of Proposition 2.14. First, by using Lemma 5.2, we see that the two functors $\mathbf{\Theta}_s$ and $\mathbf{\Theta}_t$ are exact and descent to the Grothendieck group.

Next – using (2.12) – we get for $\mathrm{i} \in \mathrm{S}$ that

$$\mathbf{\Theta}_s\mathbf{\Theta}_s(P_{\mathrm{i}}) \cong P_{\mathrm{i}}\{-2\} \oplus P_{\mathrm{i}} \oplus P_{\mathrm{i}} \oplus P_{\mathrm{i}}\{+2\} \cong \mathbf{\Theta}_s(P_{\mathrm{i}})\{-1\} \oplus \mathbf{\Theta}_s(P_{\mathrm{i}})\{+1\},$$
$$\mathbf{\Theta}_t\mathbf{\Theta}_t(P_{\mathrm{i}}) \cong \bigoplus_{\mathrm{i}-\bar{\mathrm{j}}} P_{\bar{\mathrm{j}}}\{-1\} \oplus P_{\bar{\mathrm{j}}}\{+1\} \cong \mathbf{\Theta}_t(P_{\mathrm{i}})\{-1\} \oplus \mathbf{\Theta}_t(P_{\mathrm{i}})\{+1\}.$$

We get a similar result for $P_{\bar{\mathrm{j}}}$, with $\bar{\mathrm{j}} \in \overline{\mathrm{T}}$. Therefore, we get the following natural isomorphisms of degree-preserving functors:

$$\mathbf{\Theta}_s\mathbf{\Theta}_s \cong \mathbf{\Theta}_s\{-1\} \oplus \mathbf{\Theta}_s\{+1\}, \qquad \mathbf{\Theta}_t\mathbf{\Theta}_t \cong \mathbf{\Theta}_t\{-1\} \oplus \mathbf{\Theta}_t\{+1\}.$$

Since these are the defining relations of H_∞ from (2.2), we see that $[\mathbf{G}^{\mathrm{gr}}]_{\mathbb{C}(\mathrm{v})}$ is indeed an H_∞-module. Next, choosing the evident basis of $[\mathbf{G}^{\mathrm{gr}}]_{\mathbb{C}(\mathrm{v})}$ given by the $[P_{\mathrm{i}}]$'s and using (2.12), one can see that $[\mathbf{\Theta}_s]$ and $[\mathbf{\Theta}_t]$ act as in (2.13). This shows that ζ_{G} is an H_∞-homomorphism. That ζ_{G} is bijective is clear. ■

Proof of Proposition 2.15. The claim basically follows by observing that the scalars from (2.1) for the defining relations of the θ_s and θ_t generators of H are the coefficients of the (normalized) Chebyshev polynomials (of the second kind) given by

$$\tilde{U}_0 = 1, \quad \tilde{U}_1 = X, \quad \tilde{U}_{k+1} = X\,\tilde{U}_k - \tilde{U}_{k-1} \text{ for } k \in \mathbb{Z}_{\geq 1}.$$

To give a few more details: Given a polynomial in X, one can obtain a non-commutative polynomial in two variables – say θ_s and θ_t – by replacing X^k with an alternating string $\dots\theta_s\theta_t\theta_s$ of length k (always having θ_s to the right). We write $\tilde{U}_k(\theta_s, \theta_t)$ for the non-commutative polynomial obtained from $\tilde{U}_k$ in this way. Then (2.1) implies that $\theta_{\overline{s_k}}$ is $\tilde{U}_k(\theta_s, \theta_t)$.

Now, if a representation of H_∞ factors through H_n and is annihilated by θ_{w_0}, then

$$\tilde{U}_{n-1}(\theta_s, \theta_t) = 0 = \tilde{U}_{n-1}(\theta_t, \theta_s).$$

It is not hard to deduce from this that the eigenvalues of the matrices associated to θ_s and θ_t in the representations from Definition 2.13 (these are – up to base change – of the form $\begin{pmatrix} [2]_{\mathrm{v}}^{|\mathrm{S}|} & A \\ 0 & 0 \end{pmatrix}$ respectively $\begin{pmatrix} 0 & 0 \\ A^{\mathrm{T}} & [2]_{\mathrm{v}}^{|\overline{\mathrm{T}}|} \end{pmatrix}$) have to be (multi)subsets of the (multi)subset of eigenvalues of $X\,\tilde{U}_{n-1}$. This follows from (2.6) and the observation stated after it.

The (normalized) Chebyshev polynomials have only real roots which are given by $S_{\mathrm{A}_m} \subset]-2,2[$ in (5.1) below. Now, it follows from [Smi70, Theorem 2] that the largest eigenvalue of AA^{T} or $A^{\mathrm{T}}A$ is strictly less than 4 if and only if G is as in (A), (D) or (E). (See also [BH12, Theorem 3.1.3] for a more recent proof using Perron–Frobenius theory.) This shows – keeping (2.6) in mind – that only a G as in (A), (D) or (E) (if non-trivial) can give a well-defined action of H_n.

Moreover, we have the following spectra:

$$\begin{aligned}
S_{\mathrm{A}_m} &= \left\{2\cos\left({}^{k\pi}/_{m+1}\right) \middle| k = 1, \dots, m\right\}, \\
S_{\mathrm{D}_m} &= \left\{2\cos\left({}^{k\pi}/_{2m-2}\right) \middle| k = 1, 3, 5, \dots, 2m-7, 2m-5, 2m-3\right\} \cup \{0\}, \\
S_{\mathrm{E}_6} &= \left\{2\cos\left({}^{k\pi}/_{12}\right) \middle| k = 1, 4, 5, 7, 8, 11\right\}, \\
S_{\mathrm{E}_7} &= \left\{2\cos\left({}^{k\pi}/_{18}\right) \middle| k = 1, 5, 7, 9, 11, 13, 17\right\}, \\
S_{\mathrm{E}_8} &= \left\{2\cos\left({}^{k\pi}/_{30}\right) \middle| k = 1, 7, 11, 13, 17, 19, 23, 29\right\}.
\end{aligned} \tag{5.1}$$

This list is known, see e.g. [BH12, Section 3.1.1]. The type A_m spectrum is the (multi)set of roots of the polynomial $\tilde{U}_m$. The fact that all ADE type graphs give well-defined actions of H_n then follows easily using these spectra.

In case θ_{w_0} does not act as zero the explicit form of the matrices associated to θ_s and θ_t immediately shows that any well-defined action of H_n has to be trivial, i.e. both – θ_s and θ_t – have to act as zero. ■

Remark 5.3. In Example 2.17 we have seen that the largest eigenvalue of AA^{T} and $A^{\mathrm{T}}A$ in type $\tilde{\mathrm{A}}_3$ was 4. This is true for all affine types $\tilde{\mathrm{A}}$, $\tilde{\mathrm{D}}$ and $\tilde{\mathrm{E}}$ and these are also the only graphs with this property, see e.g. [BH12, Section 3.1.1]. ▲

The following is now a direct consequence of Proposition 2.15. (Hereby $[\boldsymbol{\Theta}_{w_0}]$ corresponds to θ_{w_0} under ζ_{G} from (2.14).)

Corollary 5.4. In the setup from Section 2.3: $[\boldsymbol{\Theta}_{w_0}] = 0$ if and only if G is as in (A), (D) or (E) (if non-trivial). ■

We finish this section with the proof of Lemma 4.21, because its proof is very much in the spirit of the proof of Proposition 2.15 above.

Proof of Lemma 4.21. First note that Perron–Frobenius theory guarantees an eigenvector $\vec{\lambda}_\alpha$ of $A(G)$ with strictly positive entries. Moreover, the corresponding (so-called) Perron–Frobenius eigenvalue α is strictly positive.

Thus, after letting q to be a solution of the equation $-[2]_{\mathrm{q}} = \alpha$, we get a solution to (4.21), i.e. a $\vec{\lambda}_\alpha$ without zero entries and a $\mathrm{q} \in \mathbb{C} - \{0\}$ such that (4.21) holds. ■

Remark 5.5. Note that the proof of Lemma 4.21 is constructive and can be used to produce (4.BF2)-weightings for any given bipartite graph.

However, for ADE type graphs the Perron–Frobenius eigenvalue does not give $\mathrm{q} = \exp(\pi i/n)$, but rather $\mathrm{q} = -\exp(\pi i/n)$. For example, for $n = 3$ and type A_2 the Perron–Frobenius eigenvalue gives $\mathrm{q} = -\exp(\pi i/3)$, for which $-[2]_{\mathrm{q}} = 1$, rather than $\mathrm{q} = \exp(\pi i/3)$, for which $-[2]_{\mathrm{q}} = -1$. ▲

Since the Kazhdan–Lusztig basis elements of H_∞ are categorified by indecomposable 1-morphisms in $\mathbf{Kar}(\boldsymbol{\mathcal{D}}^\star_\infty)$, see [Eli16, Theorem 5.29], we get a stronger version of Corollary 5.4 by Theorem I later on. Namely, the matrices associated to $[\boldsymbol{\Theta}_w]$ have non-negative entries for all $w \in \mathrm{W}_\infty$ if and only if G is not of type ADE. The "if" part of this statement follows from Corollary 5.4 and the Chebyshev recursion; the "only if" part needs Theorem I. (Note hereby that for G being of type ADE we need the quantum parameter to be a root of unity to have a well-defined 2-functor and [Eli16, Theorem 5.29] does not apply anymore. Hence, there is no contradiction to Theorem I.)

5.2. **The infinite case.** First, we need to check that the maps specified in Section 4.2 are actually QG-bimodule maps and give rise to natural transformations.

Lemma 5.6. The maps from (4.d1) to (4.t2) are QG-bimodule maps. □

Proof. The map from (4.d1) is the (scaled) multiplication map and thus, a QG-bimodule map. The two maps from (4.t1) and (4.t2) clearly intertwine the left and right action of QG since their definition only involves the middle tensor factors in a non-trivial way. That the map from (4.d2) is a QG-bimodule map can be checked via a case-by-case calculation. We illustrate this in an example. To this end, let us denote the sea-green version of it by $\mathfrak{d}^*$. Then, recalling (2.QG3), we get

$$\mathfrak{d}^*(\bar{\mathrm{j}}|\underline{\mathrm{i}}) = \mathfrak{d}^*(\bar{\mathrm{j}}|\underline{\mathrm{i}}\bullet\underline{\mathrm{i}}) = \bar{\mathrm{j}}|\underline{\mathrm{i}}\bullet\mathfrak{d}^*(\underline{\mathrm{i}}) = \lambda_{\underline{\mathrm{i}}}\cdot\bar{\mathrm{j}}|\underline{\mathrm{i}}\bullet(\underline{\mathrm{i}}\otimes\underline{\mathrm{i}}|\underline{\mathrm{i}}+\underline{\mathrm{i}}|\underline{\mathrm{i}}\otimes\underline{\mathrm{i}}) = \lambda_{\underline{\mathrm{i}}}\cdot\bar{\mathrm{j}}|\underline{\mathrm{i}}\otimes\underline{\mathrm{i}}|\underline{\mathrm{i}},$$

$$\mathfrak{d}^*(\bar{\mathrm{j}}|\underline{\mathrm{i}}) = \mathfrak{d}^*(\bar{\mathrm{j}}\bullet\bar{\mathrm{j}}|\underline{\mathrm{i}}) = \mathfrak{d}^*(\bar{\mathrm{j}})\bullet\bar{\mathrm{j}}|\underline{\mathrm{i}} = \oplus_{\underline{\mathrm{i}}-\bar{\mathrm{j}}'}\lambda_{\underline{\mathrm{i}}}\cdot(\bar{\mathrm{j}}'|\underline{\mathrm{i}}\otimes\underline{\mathrm{i}}|\bar{\mathrm{j}}')\bullet\bar{\mathrm{j}}|\underline{\mathrm{i}} \overset{(2.\mathrm{QG3})}{=} \lambda_{\underline{\mathrm{i}}}\cdot\bar{\mathrm{j}}|\underline{\mathrm{i}}\otimes\underline{\mathrm{i}}|\underline{\mathrm{i}}.$$

The remaining cases can be checked verbatim. ■

The next lemmas follow directly from the definitions respectively via direct computation, and the proofs are omitted.

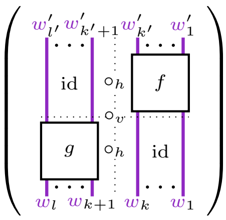
Lemma 5.7. The maps from (4.d1) to (4.t2) induce 2-morphisms in p$\mathbf{End}^{\star}(\mathbf{G}^{\mathrm{gr}})$. Moreover, the extension of the local assignment to arbitrary Soergel diagrams is consistent with the 2-structure of p$\mathbf{End}^{\star}(\mathbf{G}^{\mathrm{gr}})$. ■

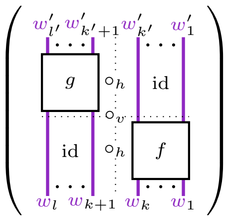
Lemma 5.8. The weightings from Definition 4.16 are (4.BF2)-weightings. ■

Proof of Theorem I, part (a). We first note that, by Lemma 5.1, Lemma 5.6 and Lemma 5.7, the weak 2-functor $\boldsymbol{\mathcal{G}}$, if well-defined, is between the stated 2-categories.

Furthermore, if $\boldsymbol{\mathcal{G}}$ is well-defined, then it extends uniquely to the Karoubi envelope by e.g. [Bor94, Proposition 6.5.9], and the corresponding diagram will commute by Proposition 2.14 (note that $[\mathbf{G}^{\mathrm{gr}}]_{\mathbb{C}(\mathrm{v})} \cong \mathrm{G}^{\mathrm{v}}$). Moreover, $\boldsymbol{\mathcal{G}}$ – if well-defined – clearly preserves all additional structures. Thus, it remains to check that $\boldsymbol{\mathcal{G}}$ is well-defined which amounts to checking that $\boldsymbol{\mathcal{G}}$ preserves the relations of $\boldsymbol{\mathcal{D}}^{\star}$.

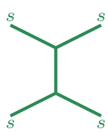
We start with the far-commutativity (4.EH), which is just the interchange law (il) in p$\mathbf{End}^{\star}(\mathbf{G}^{\mathrm{gr}})$. Denote by $\mathfrak{id}$, $\mathfrak{f}$ and $\mathfrak{g}$ the images under $\boldsymbol{\mathcal{G}}$ of the Soergel diagrams in question. Then (by our conventions how to apply $\boldsymbol{\mathcal{G}}$ to arbitrary Soergel diagrams):

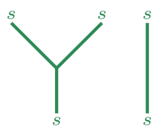
$\boldsymbol{\mathcal{G}}$  $= (\mathfrak{id} \circ_h \mathfrak{f}) \circ (\mathfrak{g} \circ_h \mathfrak{id}) \overset{\mathrm{il}}{=} (\mathfrak{id} \circ \mathfrak{g}) \circ_h (\mathfrak{f} \circ \mathfrak{id})$

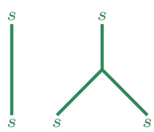
$= (\mathfrak{g} \circ \mathfrak{id}) \circ_h (\mathfrak{id} \circ \mathfrak{f}) \overset{\mathrm{il}}{=} (\mathfrak{g} \circ_h \mathfrak{id}) \circ (\mathfrak{id} \circ_h \mathfrak{f}) = \boldsymbol{\mathcal{G}}$  .

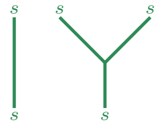
Next, we check that the other relations hold. There are several cases depending on whether i is in $\underline{\mathrm{S}}$ or $\overline{\mathrm{T}}$, as well as on the color of the involved Soergel diagrams. We do some of them and leave the other (completely similar) cases to the reader. We also omit to indicate the sources and targets of the maps. As usual, we write $x_{\underline{\mathrm{i}}} \in P_{\underline{\mathrm{i}}}$, ${}_{\underline{\mathrm{i}}}y \in {}_{\underline{\mathrm{i}}}P$ and ${}_{\underline{\mathrm{i}}}z_{\underline{\mathrm{i}}} \in {}_{\underline{\mathrm{i}}}P_{\underline{\mathrm{i}}}$. We also calculate the assignments on the corresponding direct summands only.

The Frobenius relation (4.Fr1). We get the QG-bimodule maps

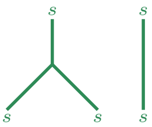
 $\overset{\boldsymbol{\mathcal{G}}}{\rightsquigarrow} \begin{cases} x_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}y & \mapsto 0, \\ x_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}}|\underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}y & \mapsto \lambda_{\underline{\mathrm{i}}}^{-1} \cdot x_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}y, \end{cases}$

 $\overset{\boldsymbol{\mathcal{G}}}{\rightsquigarrow} x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}z_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \mapsto x_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}z_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y,$

 $\overset{\boldsymbol{\mathcal{G}}}{\rightsquigarrow} \begin{cases} x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}z_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}y & \mapsto 0, \\ x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}z_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}}|\underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}y & \mapsto \lambda_{\underline{\mathrm{i}}}^{-1} \cdot x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}z_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y, \end{cases}$



$$\overset{\mathcal{G}}{\rightsquigarrow} x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}z_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \mapsto x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}z_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}y,$$



$$\overset{\mathcal{G}}{\rightsquigarrow} \begin{cases} x_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}z_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y & \mapsto 0, \\ x_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}}|\underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}z_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y & \mapsto \lambda_{\underline{\mathrm{i}}}^{-1} \cdot x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}z_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y. \end{cases}$$

These compose as claimed.

The Frobenius relation (4.Fr2). We get (keeping (2.QG3) in mind which kills a lot of terms):

$$\overset{\mathcal{G}}{\rightsquigarrow} x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \mapsto \begin{aligned} &\lambda_{\underline{\mathrm{i}}} \cdot \big(x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \bullet \underline{\mathrm{i}} \otimes \underline{\mathrm{i}}|\underline{\mathrm{i}} + x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \bullet \underline{\mathrm{i}}|\underline{\mathrm{i}} \otimes \underline{\mathrm{i}}\big) \\ &+ \textstyle\sum_{\underline{\mathrm{i}}-\bar{\mathrm{j}}} (\lambda_{\underline{\mathrm{i}}} \cdot x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \bullet \bar{\mathrm{j}}|\underline{\mathrm{i}} \otimes \underline{\mathrm{i}}|\bar{\mathrm{j}}), \end{aligned}$$

$$\overset{\mathcal{G}}{\rightsquigarrow} x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \mapsto \begin{aligned} &\lambda_{\underline{\mathrm{i}}} \cdot \big(\underline{\mathrm{i}} \otimes \underline{\mathrm{i}}|\underline{\mathrm{i}} \bullet x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y + \underline{\mathrm{i}}|\underline{\mathrm{i}} \otimes \underline{\mathrm{i}} \bullet x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y\big) \\ &+ \textstyle\sum_{\underline{\mathrm{i}}-\bar{\mathrm{j}}} (\lambda_{\underline{\mathrm{i}}} \cdot \bar{\mathrm{j}}|\underline{\mathrm{i}} \otimes \underline{\mathrm{i}}|\bar{\mathrm{j}} \bullet x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y). \end{aligned}$$

The remaining two assignments were already calculated in Example 4.13. These compose with the assignments for the trivalent vertices from (4.t1) and (4.t2) as claimed. (This relies on (2.QG3).) Let us stress that the (4.BF2)-weighting scalars cancel, since the assignments from (4.d2) and (4.t2) are multiplied by inverse scalars.

The needle relation (4.Ne). Directly by composing (4.t2) and (4.t1) (in this order).

The first barbell forcing relation (4.BF1). Using (4.d1) and (4.d2), we get:

(5.2)

$$\overset{\mathcal{G}}{\rightsquigarrow} \begin{cases} \underline{\mathrm{i}} \mapsto \lambda_{\underline{\mathrm{i}}} \cdot 2 \cdot \underline{\mathrm{i}}|\underline{\mathrm{i}}, \\ \bar{\mathrm{j}} \mapsto -[2]_{\mathrm{q}} \cdot \lambda_{\bar{\mathrm{j}}} \cdot \bar{\mathrm{j}}|\bar{\mathrm{j}}, \\ \underline{\mathrm{i}}|\underline{\mathrm{i}} \text{ and } \underline{\mathrm{i}}|\bar{\mathrm{j}} \text{ and } \bar{\mathrm{j}}|\underline{\mathrm{i}} \text{ and } \bar{\mathrm{j}}|\bar{\mathrm{j}} \mapsto 0, \end{cases}$$

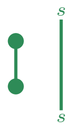
where we used (4.22) to replace $\sum_{\underline{\mathrm{i}}-\bar{\mathrm{j}}} \lambda_{\underline{\mathrm{i}}}$ by $-[2]_{\mathrm{q}} \cdot \lambda_{\bar{\mathrm{j}}}$. Next, we have

(5.3)

$$\overset{\mathcal{G}}{\rightsquigarrow} x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \mapsto \begin{aligned} &\lambda_{\underline{\mathrm{i}}} \cdot \big((x_{\underline{\mathrm{i}}} \bullet \underline{\mathrm{i}}) \otimes (\underline{\mathrm{i}}|\underline{\mathrm{i}} \bullet {}_{\underline{\mathrm{i}}}y) + (x_{\underline{\mathrm{i}}} \bullet \underline{\mathrm{i}}|\underline{\mathrm{i}}) \otimes (\underline{\mathrm{i}} \bullet {}_{\underline{\mathrm{i}}}y)\big) \\ &+ \textstyle\sum_{\underline{\mathrm{i}}-\bar{\mathrm{j}}} (\lambda_{\underline{\mathrm{i}}} \cdot (x_{\underline{\mathrm{i}}} \bullet \bar{\mathrm{j}}|\underline{\mathrm{i}}) \otimes (\underline{\mathrm{i}}|\bar{\mathrm{j}} \bullet {}_{\underline{\mathrm{i}}}y)). \end{aligned}$$

Therefore – by (5.2) – we have

(5.4)



$$\overset{\mathcal{G}}{\rightsquigarrow} x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \mapsto \begin{aligned} &\lambda_{\underline{\mathrm{i}}} \cdot \big(2 \cdot (\underline{\mathrm{i}}|\underline{\mathrm{i}} \bullet x_{\underline{\mathrm{i}}}) \otimes {}_{\underline{\mathrm{i}}}y\big) \\ &- [2]_{\mathrm{q}} \cdot \lambda_{\bar{\mathrm{j}}} \cdot ((\bar{\mathrm{j}}|\bar{\mathrm{j}} \bullet x_{\underline{\mathrm{i}}}) \otimes {}_{\underline{\mathrm{i}}}y), \end{aligned}$$

$$\overset{\mathcal{G}}{\rightsquigarrow} x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \mapsto \begin{aligned} &\lambda_{\underline{\mathrm{i}}} \cdot \big(2 \cdot x_{\underline{\mathrm{i}}} \otimes ({}_{\underline{\mathrm{i}}}y \bullet \underline{\mathrm{i}}|\underline{\mathrm{i}})\big) \\ &- [2]_{\mathrm{q}} \cdot \lambda_{\bar{\mathrm{j}}} \cdot (x_{\underline{\mathrm{i}}} \otimes ({}_{\underline{\mathrm{i}}}y \bullet \bar{\mathrm{j}}|\bar{\mathrm{j}})). \end{aligned}$$

Note that, for all $x_{\underline{\mathrm{i}}} \in P_{\underline{\mathrm{i}}}$ and all ${}_{\underline{\mathrm{i}}}y \in {}_{\underline{\mathrm{i}}}P$, we have $\bar{\mathrm{j}}|\bar{\mathrm{j}} \bullet x_{\underline{\mathrm{i}}} = 0 = {}_{\underline{\mathrm{i}}}y \bullet \bar{\mathrm{j}}|\bar{\mathrm{j}}$. This holds since all paths in $P_{\underline{\mathrm{i}}}$ start in $\underline{\mathrm{i}}$ and all paths in ${}_{\underline{\mathrm{i}}}P$ end in $\underline{\mathrm{i}}$, and $\bar{\mathrm{j}}|\bar{\mathrm{j}}$ composes only with $\bar{\mathrm{j}}$ to a non-zero element. Moreover, we also have $x_{\underline{\mathrm{i}}} \bullet \bar{\mathrm{j}}|\underline{\mathrm{i}} = 0 = \underline{\mathrm{i}}|\bar{\mathrm{j}} \bullet {}_{\underline{\mathrm{i}}}y$. In total, all the terms involving tomato t vanish. For the rest one checks directly on all possible $x_{\underline{\mathrm{i}}}, {}_{\underline{\mathrm{i}}}y$ that the claimed equation holds.

For example, in case $x_{\underline{\mathrm{i}}} = \underline{\mathrm{i}}$ and ${}_{\underline{\mathrm{i}}}y = \underline{\mathrm{i}}|\underline{\mathrm{i}}$ one gets $\lambda_{\underline{\mathrm{i}}} \cdot 2 \cdot \underline{\mathrm{i}}|\underline{\mathrm{i}} \otimes \underline{\mathrm{i}}|\underline{\mathrm{i}}$ from the sum of the two maps in (5.4), and $\lambda_{\underline{\mathrm{i}}} \cdot \underline{\mathrm{i}}|\underline{\mathrm{i}} \otimes \underline{\mathrm{i}}|\underline{\mathrm{i}}$ from the map in (5.3).

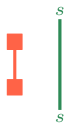
The second barbell forcing relation (4.BF2). Now the scaling will be crucial.

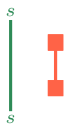
Similarly to (5.2) we get

(5.5)

$$\overset{\mathcal{G}}{\rightsquigarrow} \begin{cases} \overline{\mathrm{j}} \mapsto \lambda_{\overline{\mathrm{j}}} \cdot 2 \cdot \overline{\mathrm{j}}|\overline{\mathrm{j}}, \\ \underline{\mathrm{i}} \mapsto -[2]_{\mathrm{q}} \cdot \lambda_{\underline{\mathrm{i}}} \cdot \underline{\mathrm{i}}|\underline{\mathrm{i}}, \\ \overline{\mathrm{j}}|\overline{\mathrm{j}} \text{ and } \overline{\mathrm{j}}|\underline{\mathrm{i}} \text{ and } \underline{\mathrm{i}}|\overline{\mathrm{j}} \text{ and } \underline{\mathrm{i}}|\underline{\mathrm{i}} \mapsto 0, \end{cases}$$

The two not yet computed assignments are

(5.6)

$$\overset{\mathcal{G}}{\rightsquigarrow} x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \mapsto \begin{aligned} &\lambda_{\overline{\mathrm{j}}} \cdot \big(2 \cdot (\overline{\mathrm{j}}|\overline{\mathrm{j}} \bullet x_{\underline{\mathrm{i}}}) \otimes {}_{\underline{\mathrm{i}}}y\big) \\ &- [2]_{\mathrm{q}} \cdot \lambda_{\underline{\mathrm{i}}} \cdot \big((\underline{\mathrm{i}}|\underline{\mathrm{i}} \bullet x_{\underline{\mathrm{i}}}) \otimes {}_{\underline{\mathrm{i}}}y\big), \end{aligned}$$


$$\overset{\mathcal{G}}{\rightsquigarrow} x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \mapsto \begin{aligned} &\lambda_{\overline{\mathrm{j}}} \cdot \big(2 \cdot x_{\underline{\mathrm{i}}} \otimes ({}_{\underline{\mathrm{i}}}y \bullet \overline{\mathrm{j}}|\overline{\mathrm{j}})\big) \\ &- [2]_{\mathrm{q}} \cdot \lambda_{\underline{\mathrm{i}}} \cdot \big(x_{\underline{\mathrm{i}}} \otimes ({}_{\underline{\mathrm{i}}}y \bullet \underline{\mathrm{i}}|\underline{\mathrm{i}})\big). \end{aligned}$$

As before we have $\overline{\mathrm{j}}|\overline{\mathrm{j}} \bullet x_{\underline{\mathrm{i}}} = 0 = {}_{\underline{\mathrm{i}}}y \bullet \overline{\mathrm{j}}|\overline{\mathrm{j}}$ and the factors with the 2 die. That the claimed equation holds can be verified by a case-by-case check. (Hereby we stress that Lemma 5.8 comes crucially into the game since it ensures that the scalars add up as they should.)

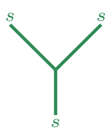
All together this shows that $\mathcal{G}$ is well-defined. ∎

Note that in the proof above we never used that G is of ADE type, but rather that we have a (4.BF2)-weighting. Thus, the above goes through – mutatis mutandis – for any bipartite graph with a (4.BF2)-weighting.

Now we switch to $\mathbb{C}$ since we use some spectral theory below.

Proof of Lemma 4.18. Assume that we have fixed G of ADE type with the corresponding root of unity q, its double-quiver algebra $\mathrm{Q}G$ and its module category $\mathbf{G}^{\mathrm{gr}}$. Assume also that we have a (well-defined) weak 2-functor $\mathcal{H}\colon \mathcal{D}_{\infty}^{\star} \to \mathrm{p}\mathbf{End}^{\star}(\mathbf{G}^{\mathrm{gr}})$ which on 1-morphisms is equal to the weak 2-functor $\mathcal{G}$ from Section 4.2.

Next, recall the bases of the left and the right $\mathrm{Q}G$-modules P_{i} and ${}_{\mathrm{i}}P$, as exemplified in (2.10). Using these bases, one easily sees that the image of the first of the two trivalent vertices has to be given by (using the notational conventions from above)


$$\overset{\mathcal{H}}{\rightsquigarrow} x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \mapsto \lambda_{\underline{\mathrm{i}}}(\mathsf{Y}) \cdot x_{\underline{\mathrm{i}}} \otimes \underline{\mathrm{i}} \otimes {}_{\underline{\mathrm{i}}}y,$$

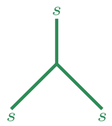
for some scalar $\lambda_{\underline{\mathrm{i}}}(\mathsf{Y}) \in \mathbb{C}$. This follows directly for degree reasons and the fact that the assignment should be a $\mathrm{Q}G$-bimodule map.

For the same reasons, the first of the two dots has to be mapped to


$$\overset{\mathcal{H}}{\rightsquigarrow} x_{\underline{\mathrm{i}}} \otimes {}_{\underline{\mathrm{i}}}y \mapsto \lambda_{\underline{\mathrm{i}}}(\uparrow) \cdot x_{\underline{\mathrm{i}}} \bullet {}_{\underline{\mathrm{i}}}y,$$

for some scalar $\lambda_{\underline{\mathrm{i}}}(\uparrow) \in \mathbb{C}$. Now, under the assumption that $\mathcal{H}$ is well-defined, the second Frobenius relation (4.Fr2) holds, which implies that both scalars have to be invertible and satisfy $\lambda_{\underline{\mathrm{i}}}(\uparrow) = (\lambda_{\underline{\mathrm{i}}}(\mathsf{Y}))^{-1}$.

Using the same reasoning, (and (4.Ne)), we see that the image of the other two generators has to be given by



$$\overset{\mathcal{H}}{\rightsquigarrow}\begin{cases} x_{\underline{\mathrm{i}}}\otimes\underline{\mathrm{i}}\otimes{}_{\underline{\mathrm{i}}}y & \mapsto 0,\\ x_{\underline{\mathrm{i}}}\otimes\underline{\mathrm{i}}|\underline{\mathrm{i}}\otimes{}_{\underline{\mathrm{i}}}y & \mapsto \lambda_{\underline{\mathrm{i}}}(\curlywedge)\cdot x_{\underline{\mathrm{i}}}\otimes{}_{\underline{\mathrm{i}}}y.\end{cases}$$

$$\overset{\mathcal{H}}{\rightsquigarrow}\begin{cases} \underline{\mathrm{i}}\mapsto\lambda_{\underline{\mathrm{i}}}(\downarrow)\cdot(\underline{\mathrm{i}}\otimes\underline{\mathrm{i}}|\underline{\mathrm{i}}+\underline{\mathrm{i}}|\underline{\mathrm{i}}\otimes\underline{\mathrm{i}}),\\ \overline{\mathrm{j}}\mapsto\sum_{\underline{\mathrm{i}}-\overline{\mathrm{j}}}(\lambda_{\underline{\mathrm{i}}}(\downarrow)\cdot\overline{\mathrm{j}}|\underline{\mathrm{i}}\otimes\underline{\mathrm{i}}|\overline{\mathrm{j}}),\end{cases}$$

for some invertible scalars $\lambda_{\underline{\mathrm{i}}}(\curlywedge),\lambda_{\underline{\mathrm{i}}}(\downarrow)\in\mathbb{C}$ satisfying $\lambda_{\underline{\mathrm{i}}}(\downarrow)=(\lambda_{\underline{\mathrm{i}}}(\curlywedge))^{-1}$.

Similarly for tomato t, where we get four invertible complex scalars which satisfy $\lambda_{\overline{\mathrm{j}}}(\uparrow)=(\lambda_{\overline{\mathrm{j}}}(\curlyvee))^{-1}$ and $\lambda_{\overline{\mathrm{j}}}(\downarrow)=(\lambda_{\overline{\mathrm{j}}}(\curlywedge))^{-1}$.

It remains to check that there is no choice for the weighting.

To this end, we write $\lambda_{\underline{\mathrm{i}}}(\updownarrow)=\lambda_{\underline{\mathrm{i}}}(\uparrow)\cdot\lambda_{\underline{\mathrm{i}}}(\downarrow)$ and $\lambda_{\overline{\mathrm{j}}}(\updownarrow)=\lambda_{\overline{\mathrm{j}}}(\uparrow)\cdot\lambda_{\overline{\mathrm{j}}}(\downarrow)$. As in the proof of Theorem I, part (a), we get

$$\overset{\mathcal{H}}{\rightsquigarrow}\begin{cases} \underline{\mathrm{i}}\mapsto\lambda_{\underline{\mathrm{i}}}(\updownarrow)\cdot 2\cdot\underline{\mathrm{i}}|\underline{\mathrm{i}},\\ \overline{\mathrm{j}}\mapsto(\sum_{\underline{\mathrm{i}}-\overline{\mathrm{j}}}\lambda_{\underline{\mathrm{i}}}(\updownarrow))\cdot\overline{\mathrm{j}}|\overline{\mathrm{j}},\\ \underline{\mathrm{i}}|\underline{\mathrm{i}}\text{ and }\underline{\mathrm{i}}|\overline{\mathrm{j}}\text{ and }\overline{\mathrm{j}}|\underline{\mathrm{i}}\text{ and }\overline{\mathrm{j}}|\overline{\mathrm{j}}\mapsto 0,\end{cases}$$

$$\overset{\mathcal{H}}{\rightsquigarrow}\begin{cases} \overline{\mathrm{j}}\mapsto\lambda_{\overline{\mathrm{j}}}(\updownarrow)\cdot 2\cdot\overline{\mathrm{j}}|\overline{\mathrm{j}},\\ \underline{\mathrm{i}}\mapsto(\sum_{\overline{\mathrm{j}}-\underline{\mathrm{i}}}\lambda_{\overline{\mathrm{j}}}(\updownarrow))\cdot\underline{\mathrm{i}}|\underline{\mathrm{i}},\\ \overline{\mathrm{j}}|\overline{\mathrm{j}}\text{ and }\overline{\mathrm{j}}|\underline{\mathrm{i}}\text{ and }\underline{\mathrm{i}}|\overline{\mathrm{j}}\text{ and }\underline{\mathrm{i}}|\underline{\mathrm{i}}\mapsto 0.\end{cases}$$

This gives the scaled versions of (5.3) and (5.6).

Using the assumption that $\mathcal{H}$ preserves the second barbell forcing relation (4.BF2), we can relate the $\lambda_{\underline{\mathrm{i}}}(\updownarrow)$'s and the $\lambda_{\overline{\mathrm{j}}}(\updownarrow)$'s. Since relation (4.BF2) holds, we see that an equation of the form (4.22) must be satisfied for the $\lambda_{\underline{\mathrm{i}}}(\updownarrow)$'s and the $\lambda_{\overline{\mathrm{j}}}(\updownarrow)$'s. Spectral theory shows that the eigenspace of $A(G)$ for the eigenvalue $-[2]_{\mathrm{q}}$ is one-dimensional for ADE type graphs and the corresponding roots of unity, cf. (5.1). This means that our chosen (4.BF2)-weighting from Definition 4.16 is unique up to a scalar ς with

$$\varsigma=\lambda_{\underline{\mathrm{i}}}(\updownarrow)\cdot\lambda_{\underline{\mathrm{i}}}^{-1}=\lambda_{\overline{\mathrm{j}}}(\updownarrow)\cdot\lambda_{\overline{\mathrm{j}}}^{-1},\quad\text{for all }\underline{\mathrm{i}},\overline{\mathrm{j}}\in G.$$

Observe that this amounts to saying that the corresponding products do not depend on the vertices.

The first part of the claim now follows by letting τ be ς divided by the value of the end dot $\lambda_{\underline{\mathrm{i}}}(\uparrow)$ and υ be ς divided by the value of the start dot $\lambda_{\underline{\mathrm{i}}}(\downarrow)$ (for any vertex).

Clearly, any consistent rescaling of the $\mathrm{Q}G$-bimodule maps associated to the generating 2-morphisms of $\mathcal{D}_{\infty}^{\star}$ gives rise to an equivalence of 2-representations. ■

Note that the one-dimensionality of the eigenspace of $A(G)$ for the eigenvalue $-[2]_{\mathrm{q}}$ fails in general for bipartite graphs. Consequently, Lemma 4.18 is not always true for bipartite graphs with a given (4.BF2)-weighting.

5.3. **The finite case.** We again use the setup from Section 4.2.

Proof of Theorem I, part (b). By statement (a) of Theorem I – and the evident analog of Lemma 5.6 for the maps from (4.2nv) – it only remains to check three extra relations, i.e. the relations (4.2nv1) to (4.2nv3) need to be preserved under the functor $\mathcal{G}_n$. The relations

(4.2nv1) and (4.2nv3) are clearly preserved, and we need to prove that (4.2nv2) is preserved if and only if G is of ADE type.

"Only if". This is clear by Proposition 2.15, since otherwise $[\mathbf{G}^{\mathrm{gr}}]_{\mathbb{C}(\mathrm{v})}$ would inherit the structure of an H_n-module.

"If". Note that the JW_k^{u} are well-defined for $k \in \{0, \dots, n\}$, both when q is generic and when q is a complex, primitive $2n$th root of unity.

Let us assume that G is of ADE type. Below we show that $\boldsymbol{\mathcal{G}}_n(\mathrm{JW}_n^{u}) = 0$, when q is our fixed complex, primitive $2n$th root of unity. This implies, by definition, that (4.2nv2) is preserved under $\boldsymbol{\mathcal{G}}_n$.

We have (see e.g. [Kho05, Section 3] and [Eli16, Section 2.2])

$$[V_l] = \textstyle\sum_{l=0}^{k} d_{l+1}^{k+1} \cdot [V_1^{\otimes k}], \quad 0 \leq k \leq l, \tag{5.7}$$

in the Grothendieck ring of $\mathrm{U_q}(\mathfrak{sl}_2)$-**Mod** (the finite-dimensional type 1 modules of quantum $\mathfrak{sl}_2$) for q not a root of unity or generic. Here the V_l's are the simple objects of highest weight l, and the d_l^k's are as in (2.1).

The same holds in the semisimplified quotient category $\mathrm{U_q}(\mathfrak{sl}_2)$-$\mathbf{Mod}_{\mathrm{s}}$, when q is a complex, primitive $2n$th root of unity as long as $l \leq n$. (See e.g. [And03, Theorem 3.1] for a version that works for even roots of unity. See also e.g. [Saw06, Section 5] for the semisimplified quotient.) Recall also that $[V_n] = 0$ holds in $[\mathrm{U_q}(\mathfrak{sl}_2)\text{-}\mathbf{Mod}_{\mathrm{s}}]_{\mathbb{C}(\mathrm{v})}$.

Note that the sign of d_{l+1}^{k+1} alternates as k runs from 0 to l. Khovanov [Kho05, Theorem 1] showed that the right-hand side of (5.7) is the Euler characteristic of a complex

$$C_l^* \colon C_l^0 \to C_l^2 \to \cdots \to C_l^{l/2}, \text{ for } l \text{ even}, \; C_l^* \colon C_l^0 \to C_l^2 \to \cdots \to C_l^{l-1/2}, \text{ for } l \text{ odd},$$

which is acyclic in positive cohomological degree and $H^0(C_l^*) \cong V_l$. More explicitly,

$$C_l^k = \left(V_1^{\otimes(l-2k)}\right)^{\oplus d_{l+1}^{k+1}},$$

and the differential is defined as a direct sum obtained by applying the evaluation map $V_1 \otimes V_1 \to \mathbb{C}$, multiplied by a suitable sign, to each neighboring tensor pair $V_1 \otimes V_1$ in each copy of $V_1^{\otimes(l-2k)}$.

Note that Khovanov proves his theorem for $\mathrm{q} = 1$ and mentions that the proof also works for generic q. However, his proof follows by induction from the fact that $V_1 \otimes V_l \cong V_{l-1} \otimes V_{l+1}$, which is also true in our case (by a special case of the so-called linkage principle) as long as $0 \leq l \leq n-1$.

Since $\mathrm{U_q}(\mathfrak{sl}_2)$-$\mathbf{Mod}_{\mathrm{s}}$ is a semisimple abelian category, its bounded derived category is also semisimple [GM03, Section III.3] and equivalent to $\bigoplus_{\mathbb{Z}} \mathrm{U_q}(\mathfrak{sl}_2)$-$\mathbf{Mod}_{\mathrm{s}}$ by the homology functor. Moreover, the bounded derived category is equivalent to the homotopy category of bounded complexes, because all objects are projective, see e.g. [Wei94, Theorem 10.4.8]. Thus, C_l^* is equivalent to V_l (seen as a complex concentrated in homological degree zero).

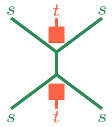
By [Eli16, Proposition 1.2], this implies that, for any $0 \leq l \leq n$, the object $(\overline{u}_l, \mathrm{JW}_l^{u})$ in $\mathbf{Kar}(\boldsymbol{\mathcal{D}})$ is homotopy equivalent to a complex $D_l^*(u)$ with

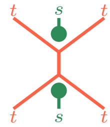
$$D_l^k(u) = \overline{u}_l^{\oplus d_{l+1}^{k+1}}, \quad \text{for } u = s, t.$$

Since this holds in the finite and infinite case alike, the definitions of the complex $D_l^*(u)$ and the homotopy equivalence with $(\overline{u}_l, \mathrm{JW}_l^{u})$ do not use any $2n$-vertices. For the same reason, the defining properties of the homotopy equivalence follow from the relations in $\boldsymbol{\mathcal{D}}$ which do not involve $2n$-vertices.

Since we already proved that $\boldsymbol{\mathcal{G}}_n$ preserves all the relations which do not involve $2n$-vertices, we see that $\boldsymbol{\mathcal{G}}_n(D_n^*(u))$ is a complex and that it is homotopy equivalent to $\boldsymbol{\mathcal{G}}_n(\overline{u}_n, \mathrm{JW}_n^{u})$. The latter has Euler characteristic equal to zero, by Proposition 2.15. Therefore, $\boldsymbol{\mathcal{G}}_n(\mathrm{JW}_n^u)=0$, which is what we had to prove. ■

Example 5.9. In principle one could also check by hand that $\boldsymbol{\mathcal{G}}_n(\mathrm{JW}_n^u)=0$ holds. This is a daunting task in general, but let us do a small case which is quite illustrative. Fix the type A_2 graph with vertices $\underline{1}$ and $\overline{2}$, and the weighting is $\lambda_{\underline{1}}=[1]_{\mathrm{q}}=1$ and $\lambda_{\overline{2}}=-[2]_{\mathrm{q}}=-1$. In this case $n=3$ and q is the complex, primitive 6th root of unity $\mathrm{q}=\exp(\pi i/3)$. After a small calculation one gets



$$\overset{\boldsymbol{\mathcal{G}}_n}{\rightsquigarrow}\lambda_{\underline{1}}^{-1}\lambda_{\overline{2}}\cdot\mathrm{ID}_X,$$


$$\overset{\boldsymbol{\mathcal{G}}_n}{\rightsquigarrow}\lambda_{\underline{1}}\lambda_{\overline{2}}^{-1}\cdot\mathrm{ID}_Y.$$

Hereby X and Y are the corresponding tensor products for the boundary sequences. Hence, with our weighting, we get $\boldsymbol{\mathcal{G}}_3(\mathrm{JW}_3^s)=0$ and $\boldsymbol{\mathcal{G}}_3(\mathrm{JW}_3^t)=0$, since the two summands in their expressions in Example 4.8 cancel. ▲

Proof of Lemma 4.19. If a 2-representation of $\boldsymbol{\mathcal{D}}_n^\star$ agrees with our $\boldsymbol{\mathcal{G}}$ on 1-morphisms, then its decategorification has to be isomorphic to $[\boldsymbol{\mathcal{G}}]$. Therefore, it has to kill $[\boldsymbol{\Theta}_{w_0}]$. But $\boldsymbol{\Theta}_{w_0}$ corresponds to the idempotent defined as the image of the 2-colored version of the "Jones–Wenzl projector" JW_n^u, so that has to be sent to zero by the above. This implies that the $2n$-valent vertex has to be sent to zero as well by [Eli16, (6.16)]. ■

5.4. **Classification of dihedral 2-representations.** We work over $\mathbb{C}$ in Theorem II.

Proof of Theorem II. We have to prove the statements (a), (b), (c) and (d).

Proof of statement (a). By Proposition 4.12, $\mathbf{Kar}(\boldsymbol{\mathcal{D}}_n)^\star$ and $\mathbf{Kar}(\boldsymbol{\mathcal{D}}_\infty)^\star$ are graded (locally) finitary, and thus, we see that we actually get graded finitary, weak 2-representations in Theorem I.

Hence, we only need to show that $\boldsymbol{\mathcal{G}}$ is simple transitive. Transitivity follows from the connectivity of G. For each P_{i} and P_{j}, apply an alternating sequence $\boldsymbol{\Theta}_{\mathrm{alt}}$ of $\boldsymbol{\Theta}_s$ and $\boldsymbol{\Theta}_t$, starting in the opposite color of $\mathrm{i}\in G$, to P_{i} of length determined by the minimal path connecting i and j in the graph G. Then $P_{\mathrm{j}}\{\mathsf{a}\}$ will be a summand of $\boldsymbol{\Theta}_{\mathrm{alt}}(P_{\mathrm{i}})$ for some shift $\mathsf{a}\in\mathbb{Z}$. This, by Remark 3.7, shows transitivity.

It remains to show that there are no non-trivial ideals. This is clearly true in case G has only one vertex. Otherwise, fix one sea-green s colored vertex $\underline{\mathrm{i}}$. Let us restrict the 2-action to $\boldsymbol{\Theta}_s$ on the additive category generated by $P_{\underline{\mathrm{i}}}$. This category is equivalent to the category of graded D-modules for $\mathrm{D}=\mathbb{C}[X]/(X^2)$. Hence, the 2-action of $\boldsymbol{\Theta}_s$ is given by tensoring with the biprojective D-bimodule $\mathrm{D}\otimes\mathrm{D}$, see e.g. [MM17, Section 3.3]. Thus, the non-identity endomorphism $\underline{\mathrm{i}}|\underline{\mathrm{i}}$ on $P_{\underline{\mathrm{i}}}$ cannot belong to any ideal which is stable under the 2-action and does not contain any identity morphism (this follows from [MM17, Proposition 2]). This, of course, holds for any vertex i, i.e. none of the maps $\mathrm{i}|\mathrm{i}$ are in such an ideal. Now, if $\underline{\mathrm{i}}|\overline{\mathrm{j}}\colon P_{\overline{\mathrm{j}}}\to P_{\underline{\mathrm{i}}}$ (or $\overline{\mathrm{j}}|\underline{\mathrm{i}}\colon P_{\underline{\mathrm{i}}}\to P_{\overline{\mathrm{j}}}$) would be contained in such an ideal, then, because it is a 2-ideal, composition with $\overline{\mathrm{j}}|\underline{\mathrm{i}}\colon P_{\underline{\mathrm{i}}}\to P_{\overline{\mathrm{j}}}$ (or $\underline{\mathrm{i}}|\overline{\mathrm{j}}\colon P_{\overline{\mathrm{j}}}\to P_{\underline{\mathrm{i}}}$) would imply that the result $\underline{\mathrm{i}}|\underline{\mathrm{i}}\colon P_{\underline{\mathrm{i}}}\to P_{\underline{\mathrm{i}}}$ (or $\overline{\mathrm{j}}|\overline{\mathrm{j}}\colon P_{\overline{\mathrm{j}}}\to P_{\overline{\mathrm{j}}}$) belongs to the ideal, which is a contradiction.

Proof of statement (b). (Below we use $'$ as a notation for anything related to G'. Similarly in the rest of the whole proof.) First assume that G and G' are isomorphic as bipartite graphs by some map $f\colon G\to G'$. Such an isomorphism induces a reordering of the indecomposables of $\mathbf{G}^{\mathrm{gr}}$ and $(\mathbf{G}')^{\mathrm{gr}}$ via $P_{\underline{\mathrm{i}}}\mapsto P'_{\underline{\mathrm{f(i)}}}$ and $P_{\overline{\mathrm{i}}}\mapsto P'_{\overline{\mathrm{f(i)}}}$. Thus, we can define a functor

$$\boldsymbol{\mathcal{F}}\colon\mathbf{G}^{\mathrm{gr}}\to(\mathbf{G}')^{\mathrm{gr}},\quad\boldsymbol{\mathcal{F}}(P_{\mathrm{i}})=P'_{\mathrm{f(i)}},$$

which maps each morphism in $\mathbf{G}^{\mathrm{gr}}$ to the evident (f-reordered) morphism in $(\mathbf{G}')^{\mathrm{gr}}$. This is well-defined since f is an isomorphism of bipartite graphs (in particular, for each $\mathrm{i} \in G$ the image $f(\mathrm{i}) \in G'$ has precisely the same two-step connectivity neighborhood). This functor is clearly structure preserving, and it is an equivalence since for all $\mathrm{i} \in G$ the valencies are preserved (which shows that the functor induces isomorphisms between the finite-dimensional hom-spaces). Thus, $\boldsymbol{\mathcal{F}}$ is a structure preserving equivalence of categories.

By construction, $\boldsymbol{\Theta}_s$ is given by tensoring with sums of $\mathbb{Q}G$-bimodules of the form $P_{\mathrm{i}}\{-1\} \otimes {}_{\mathrm{i}}P$, while $\boldsymbol{\Theta}_s'$ is given by tensoring with sums of $\mathbb{Q}G$-bimodules of the form $P_{\mathrm{i}}'\{-1\} \otimes {}_{\mathrm{i}}P'$. Thus, there is a commuting diagram

$$\begin{tikzcd}
\mathbf{G}^{\mathrm{gr}} \arrow[r, "\boldsymbol{\Theta}_s"] \arrow[d, "\boldsymbol{\mathcal{F}}"'] & \mathbf{G}^{\mathrm{gr}} \arrow[d, "\boldsymbol{\mathcal{F}}"] \\
(\mathbf{G}')^{\mathrm{gr}} \arrow[r, "\boldsymbol{\Theta}_s'"'] & (\mathbf{G}')^{\mathrm{gr}},
\end{tikzcd}$$

which similarly exists for $\boldsymbol{\Theta}_t$ and $\boldsymbol{\Theta}_t'$ as well. Hence, we have $\boldsymbol{\Theta}_s' = \boldsymbol{\mathcal{F}}\boldsymbol{\Theta}_s\boldsymbol{\mathcal{F}}^{-1}$ and $\boldsymbol{\Theta}_t' = \boldsymbol{\mathcal{F}}\boldsymbol{\Theta}_t\boldsymbol{\mathcal{F}}^{-1}$. This means that $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{G}}'$ agree on 1- and on 2-morphisms of $\boldsymbol{\mathcal{D}}^{\star}$ up to $\boldsymbol{\mathcal{F}}$-conjugation. Thus, this induces a (degree-zero) modification between $\widetilde{\boldsymbol{\mathcal{G}}}$ and $\widetilde{\boldsymbol{\mathcal{G}}}'$, which is a 2-isomorphism.

Vice versa, any equivalence between $\widetilde{\boldsymbol{\mathcal{G}}}$ and $\widetilde{\boldsymbol{\mathcal{G}}}'$ sends indecomposables to indecomposables, so it descends to an intertwining isomorphism of the form

$$[\mathbf{G}^{\mathrm{gr}}]_{\mathbb{C}(\mathrm{v})} \xrightarrow{\cong} [(\mathbf{G}')^{\mathrm{gr}}]_{\mathbb{C}(\mathrm{v})}, \quad [P_{\mathrm{i}}] \mapsto [P_{\mathrm{i}'}']$$

between the Grothendieck groups. Such an isomorphism can neither send a $[P_{\underline{\mathrm{i}}}]$ to a $[P_{\overline{\mathrm{i}'}}']$ nor a $[P_{\overline{\mathrm{i}}}]$ to a $[P_{\underline{\mathrm{i}'}}']$, since this will not intertwine the action of $[\boldsymbol{\Theta}_s]$ and $[\boldsymbol{\Theta}_t]$ to $[\boldsymbol{\Theta}_s']$ and $[\boldsymbol{\Theta}_t']$. In particular, the number of sea-green $\underline{\mathrm{j}}$ and tomato $\overline{\mathrm{j}}$ colored vertices has to be the same for G and G'. Further, for the same reasons, if such an isomorphism sends $[P_{\mathrm{i}}]$ and $[P_{\mathrm{j}}]$ to $[P_{\mathrm{i}'}']$ and $[P_{\mathrm{j}'}']$, and i, j are connected via an edge in G, then i', j' are connected via an edge in G'. In total, such a permutation gives rise to an isomorphism of bipartite graphs $f\colon G \to G'$ defined via $f(\mathrm{i}) = \mathrm{i}'$.

Proof of statement (c). First – by using traces – one can see that two bipartite graphs G and G' which are spectrum-color-inequivalent give non-isomorphic H-modules. Precisely, for all $k \in \mathbb{Z}_{\geq 0}$, we have (top: even k; bottom: odd k):

$$\begin{aligned}
&[\boldsymbol{\Theta}_{\overline{s_k}}] = (AA^{\mathrm{T}})^{\frac{k-2}{2}} \cdot \begin{pmatrix} AA^{\mathrm{T}} & A[2]_{\mathrm{v}} \\ 0 & 0 \end{pmatrix}, \; [\boldsymbol{\Theta}_{\overline{t_k}}] = (A^{\mathrm{T}}A)^{\frac{k-2}{2}} \cdot \begin{pmatrix} 0 & 0 \\ A^{\mathrm{T}}[2]_{\mathrm{v}} & A^{\mathrm{T}}A \end{pmatrix}, \\
&[\boldsymbol{\Theta}_{\overline{s_k}}] = (AA^{\mathrm{T}})^{\frac{k-1}{2}} \cdot \begin{pmatrix} [2]_{\mathrm{v}} & A \\ 0 & 0 \end{pmatrix}, \; [\boldsymbol{\Theta}_{\overline{t_k}}] = (A^{\mathrm{T}}A)^{\frac{k-1}{2}} \cdot \begin{pmatrix} 0 & 0 \\ A^{\mathrm{T}} & [2]_{\mathrm{v}} \end{pmatrix}.
\end{aligned} \tag{5.8}$$

(Where $\boldsymbol{\Theta}_{\overline{s_k}}$ and $\boldsymbol{\Theta}_{\overline{t_k}}$ mean the evident analog of the notation from Section 2.1.) Here A is as in (2.5) for $A(G)$. This works completely analogously for $[\boldsymbol{\Theta}_{\overline{s_k}}']$ and $[\boldsymbol{\Theta}_{\overline{t_k}}']$ as well, where one uses A' – coming from $A(G')$ – instead of A.

If $|\underline{\mathrm{S}}| \neq |\underline{\mathrm{S}}'|$ or $|\overline{\mathrm{T}}| \neq |\overline{\mathrm{T}}'|$, then the bottom row in (5.8) produces different traces, so the representations are non-isomorphic.

If $S_G \neq S_{G'}$, then one also gets non-isomorphic representations. To see this, note that the spectrum of the matrices in the top row associated to $[\boldsymbol{\Theta}_{\overline{s_k}}]$ and $[\boldsymbol{\Theta}_{\overline{t_k}}]$, is that of $(AA^{\mathrm{T}})^{k+2/2}$ and $(A^{\mathrm{T}}A)^{k+2/2}$, for all $k \in \mathbb{Z}_{\geq 0}$. A similar observation holds for the spectrum of $[\boldsymbol{\Theta}_{\overline{s_k}}']$ and $[\boldsymbol{\Theta}_{\overline{t_k}}']$. So these spectra consist of powers of the elements of S_G and $S_{G'}$, respectively. Since the spectrum is an invariant of the representation, the result follows.

Next, let us assume that $A(G)$ and $A(G')$ have the same spectra. Thus, by (2.5), there exists a singular value decomposition of the form

$$A = U\Sigma V^{*}, \qquad A' = W\Sigma X^{*}$$

with unitary, complex-valued matrices U, V, W, X of the appropriate sizes. (Note that we work over $\mathbb{C}(\mathrm{v})$, but $A(G)$ and $A(G')$ are integral matrices. Hence, their singular value decompositions exist.) Thus,

$$\begin{aligned}\begin{pmatrix} WU^{*} & 0 \\ 0 & XV^{*}\end{pmatrix}[\boldsymbol{\Theta}_{\textcolor{green}{s}}]\begin{pmatrix} UW^{*} & 0 \\ 0 & VX^{*}\end{pmatrix} &= \begin{pmatrix} WU^{*} & 0 \\ 0 & XV^{*}\end{pmatrix}\begin{pmatrix} [2]_{\mathrm{v}} & A \\ 0 & 0\end{pmatrix}\begin{pmatrix} UW^{*} & 0 \\ 0 & VX^{*}\end{pmatrix} \\ &= \begin{pmatrix} [2]_{\mathrm{v}} & A' \\ 0 & 0\end{pmatrix} = [\boldsymbol{\Theta}'_{\textcolor{green}{s}}],\end{aligned}$$

and similarly – using the same matrices for conjugation – for $[\boldsymbol{\Theta}_{\textcolor{red}{t}}]$ and $[\boldsymbol{\Theta}'_{\textcolor{red}{t}}]$. (Hereby we have also used that $|\textcolor{green}{\underline{\mathrm{S}}}| = |\textcolor{green}{\underline{\mathrm{S}}}'|$ and $|\textcolor{red}{\overline{\mathrm{T}}}| = |\textcolor{red}{\overline{\mathrm{T}}}'|$.)

This shows that there is a change of basis such that $[\boldsymbol{\Theta}_{\textcolor{green}{s}}]$ is sent to $[\boldsymbol{\Theta}'_{\textcolor{green}{s}}]$ and $[\boldsymbol{\Theta}_{\textcolor{red}{t}}]$ is sent to $[\boldsymbol{\Theta}'_{\textcolor{red}{t}}]$, showing that the underlying H-modules are in fact isomorphic.

Proof of statement (d). By using the relations of $\boldsymbol{\mathcal{D}}^{\star}$ one easily sees that

$$2\mathrm{End}_{\boldsymbol{\mathcal{D}}^{\star}}(\varnothing) \cong \mathbb{C}[\textcolor{green}{\updownarrow}, \textcolor{red}{\updownarrow}] = \mathrm{R}, \qquad \deg(\textcolor{green}{\updownarrow}) = \deg(\textcolor{red}{\updownarrow}) = 2,$$

where $\textcolor{green}{\updownarrow}, \textcolor{red}{\updownarrow}$ are the "floating barbells", see [Eli16, Corollary 5.20 and Lemma 6.23].

It follows from the barbell forcing relations (4.BF1) and (4.BF2) that the 2-hom spaces in $\boldsymbol{\mathcal{D}}^{\star}$ are graded R-bimodules with a generating set given by Soergel diagrams without "floating components". Hence, $\boldsymbol{\mathcal{D}}^{\star}$ is defined over the polynomial algebra R, see [Eli16, Proposition 5.19 and Proposition 6.22].

Now fix $n \in \mathbb{Z}_{>1}$ (the case $n = 1$ is discussed in Example 3.8). Let $\mathrm{R}^{\mathrm{W}_n} \subset \mathrm{R}$ denote the subalgebra of invariant elements. The coinvariant algebra is defined as

$$\mathrm{C}^{+}_{\mathrm{W}_n} = \mathrm{R}/I^{+},$$

where I^{+} is the ideal generated by the elements in $\mathrm{R}^{\mathrm{W}_n}$ which are homogeneous of positive degree. Define

$$\begin{gathered} z = \textcolor{green}{\updownarrow}\,\textcolor{green}{\updownarrow} - [2]_{\mathrm{q}} \cdot \textcolor{green}{\updownarrow}\,\textcolor{red}{\updownarrow} + \textcolor{red}{\updownarrow}\,\textcolor{red}{\updownarrow} \in \mathbb{C}[\textcolor{green}{\updownarrow}, \textcolor{red}{\updownarrow}], \\ Z = \textstyle\prod_{\textcolor{violet}{w} \in \mathrm{W}_n} \textcolor{violet}{w}(2 \cdot \textcolor{green}{\updownarrow} + [2]_{\mathrm{q}} \cdot \textcolor{red}{\updownarrow}) \in \mathbb{C}[\textcolor{green}{\updownarrow}, \textcolor{red}{\updownarrow}], \quad \text{where} \\ \textcolor{green}{s}(\textcolor{green}{\updownarrow}) = \textcolor{green}{\updownarrow}, \quad \textcolor{red}{t}(\textcolor{green}{\updownarrow}) = \textcolor{green}{\updownarrow} - [2]_{\mathrm{q}} \cdot \textcolor{red}{\updownarrow}, \quad \textcolor{green}{s}(\textcolor{red}{\updownarrow}) = \textcolor{red}{\updownarrow} - [2]_{\mathrm{q}} \cdot \textcolor{green}{\updownarrow}, \quad \textcolor{red}{t}(\textcolor{red}{\updownarrow}) = \textcolor{red}{\updownarrow}. \end{gathered}$$

(Note that the element Z does not make sense for "$n = \infty$".) By [Eli16, Claim 3.23], the subalgebra $\mathrm{R}^{\mathrm{W}_n}$ is isomorphic to $\mathbb{C}[z, Z] \subset \mathrm{R}$.

Thus, in order to check that our weak 2-functors $\boldsymbol{\mathcal{G}}_n$ descend from $\boldsymbol{\mathcal{D}}^{\star}_n$ to $(\boldsymbol{\mathcal{D}}^{\mathrm{f}}_n)^{*}$, we have to check that $\boldsymbol{\mathcal{G}}_n(z) = 0 = \boldsymbol{\mathcal{G}}_n(Z)$.

First note that any product of two or more barbells is zero, because

$$\textcolor{green}{\underline{\mathrm{i}}}|\textcolor{green}{\underline{\mathrm{i}}} \bullet \textcolor{green}{\underline{\mathrm{i}}}|\textcolor{green}{\underline{\mathrm{i}}} = \textcolor{red}{\overline{\mathrm{j}}}|\textcolor{red}{\overline{\mathrm{j}}} \bullet \textcolor{green}{\underline{\mathrm{i}}}|\textcolor{green}{\underline{\mathrm{i}}} = \textcolor{green}{\underline{\mathrm{i}}}|\textcolor{green}{\underline{\mathrm{i}}} \bullet \textcolor{red}{\overline{\mathrm{j}}}|\textcolor{red}{\overline{\mathrm{j}}} = \textcolor{red}{\overline{\mathrm{j}}}|\textcolor{red}{\overline{\mathrm{j}}} \bullet \textcolor{red}{\overline{\mathrm{j}}}|\textcolor{red}{\overline{\mathrm{j}}} = 0.$$

By (5.2) and (5.5), this implies that $\boldsymbol{\mathcal{G}}_n(z) = 0$.

The same argument shows that $\boldsymbol{\mathcal{G}}_n(Z) = 0$, e.g. we already have

$$(2 \cdot \textcolor{green}{\updownarrow} + [2]_{\mathrm{q}} \cdot \textcolor{red}{\updownarrow})\textcolor{green}{s}(2 \cdot \textcolor{green}{\updownarrow} + [2]_{\mathrm{q}} \cdot \textcolor{red}{\updownarrow}) = (2 \cdot \textcolor{green}{\updownarrow} + [2]_{\mathrm{q}} \cdot \textcolor{red}{\updownarrow})((2 - [2]_{\mathrm{q}}^{2}) \cdot \textcolor{green}{\updownarrow} + [2]_{\mathrm{q}} \cdot \textcolor{red}{\updownarrow}) = 0.$$

When "$n = \infty$", one has $\mathrm{R}^{\mathrm{W}_{\infty}} \cong \mathbb{C}[z]$. This case therefore follows in the same way as in the finite case. ∎

Example 5.10. Note that the proof of (c) of Theorem II is effective in the sense that one can explicitly compute the change of basis matrix via the singular value decomposition. For the type E_6 situation from Example 2.3 one gets for instance

$$UW^* = \frac{1}{2\sqrt{6}} \cdot \begin{pmatrix} \sqrt{6}-2 & -\sqrt{6}-2 & 2 \\ -2 & -2 & 4 \\ -\sqrt{6}-2 & \sqrt{6}-2 & 2 \end{pmatrix}, \qquad VX^* = (UW^*)^{\mathrm{T}},$$

(using the ordering of the vertices from left to right, sea-green ● before tomato ■) which gives the (highly non-integral) change of basis matrix. ▲

Proof of Theorem III. By Theorem II it remains to rule out the case that there are graded simple transitive 2-representations which are not as in Theorem I.

In order to rule these out: By (d) of Theorem II and Remark 3.9, any graded simple transitive 2-representation of $\mathbf{Kar}(\boldsymbol{\mathcal{D}}_n)^{\star}$ would give the corresponding simple transitive 2-representation of $\mathbf{Kar}(\boldsymbol{\mathcal{D}}_n^{\mathrm{f}})$. The underlying quiver of such a 2-representation is of ADE type and their action on the level of the Grothendieck groups is fixed up to change of basis, see [KMMZ16, Sections 6 and 7]. This structure on the level of 1-morphisms is preserved under strictification. By combining Lemma 4.18 and Lemma 4.19 ("uniqueness of higher structure"), these are equivalent to the ones from Theorem I. Indeed, we can define a degree-preserving autoequivalence of $\mathbf{G}^{\mathrm{gr}}$: on objects it is the identity; on morphisms it is given by multiplying any homogeneous $\mathbb{Q}G$-bimodule map by (suitable) powers of τ and υ. By definition, this autoequivalence intertwines the 2-representation with the τ,υ-scaling and the one from Theorem I. ■

Note that the proof of Theorem III relies on Lemma 4.18, whose proof (as given in this paper) uses the grading in an essential way.

Remark 5.11. In fact, in order to complete the classification in [KMMZ16] it remains to compare different choices for q, i.e. we fix G and consider two different complex, primitive $2n$th root of unity and the associated 2-representations constructed in Section 4.2. We have to show that they are equivalent when working over $\mathbb{C}$ and over Soergel bimodules (since the Soergel bimodules "do not see the q's").

Luckily, this follows immediately from [Eli16, Theorem 6.24], which shows that the 2-color Soergel calculi for different choices of q are equivalent. This equivalence gives rise to an equivalence of the two 2-representations in question. ▲

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